



This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

- + *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
- + *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
- + *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
- + *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at <http://books.google.com/>

Sci 920.51

PERIODICAL SHELVES

Harvard College Library



BOUGHT WITH THE INCOME

FROM THE BEQUEST
IN MEMORY OF

JOHN FARRAR

Hollis Professor of Mathematics, Astronomy, and
Natural Philosophy

MADE BY HIS WIDOW

ELIZA FARRAR

FOR

"BOOKS IN THE DEPARTMENT OF MATHEMATICS,
ASTRONOMY, AND NATURAL PHILOSOPHY"

SCIENCE CENTER LIBRARY

**BULLETIN OF THE
AMERICAN
MATHEMATICAL SOCIETY**

**A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE**

EDITED BY

F. N. COLE	VIRGIL SNYDER	J. W. YOUNG
ALEXANDER ZIWET		D. E. SMITH
T. LEVI-CIVITA		R. C. ARCHIBALD

VOL. XXVI

OCTOBER 1919 TO JULY 1920

**PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK
1920**

THE 90 BIRTHDAY

OF

THE NEW ERA PRINTING COMPANY

LANCASTER, PA.

PRESS OF
THE NEW ERA PRINTING COMPANY
LANCASTER, PA.

THE NEW ERA PRINTING COMPANY
LANCASTER, PA.

THE NEW ERA PRINTING COMPANY

THE NEW ERA PRINTING COMPANY

THE NEW ERA PRINTING COMPANY

THE NEW ERA PRINTING COMPANY



BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

EDITED BY

F. N. COLE	VIRGIL SNYDER	J. W. YOUNG
ALEXANDER ZIWET	D. E. SMITH	
T. LEVI-CIVITA	R. C. ARCHIBALD	

VOLUME XXVI., NUMBER 1

OCTOBER, 1919

PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK

1919

Entered at the Post Office at Lancaster, Pa., as second-class matter.
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER

Whole No. 281

Digitized by Google

*For upper-class and graduate students
in American colleges and universities*

PROJECTIVE GEOMETRY

Volume I

By **OSWALD VEBLEN**, *Princeton University*, and **J. W. YOUNG**, *Dartmouth College*.

Volume II

By **OSWALD VEBLEN**

An exposition of the principles of geometry.

Volume I treats projective geometry as a logical structure based upon explicitly stated assumptions.

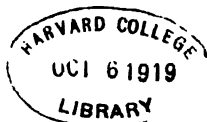
Volume II completes the program outlined in the first volume.



GINN AND COMPANY

70 Fifth Avenue

NEW YORK



BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY.

IN MEMORY OF GABRIEL MARCUS GREEN.

BY PROFESSOR E. J. WILCZYNSKI.

(Read before the American Mathematical Society March 28, 1919).

GABRIEL MARCUS GREEN was born in the city of New York on October 19, 1891. He died on January 24, 1919, at Cambridge, Massachusetts. In this brief span of years he has won enduring fame. In spite of his extreme youth, we mourn in him not the promise of a genius unfulfilled, but the sad untimely loss of a great leader of proven strength whose power and insight had been fully tested, and whose actual achievements can never perish.

Green graduated from Public School No. 4 of New York in 1904 as valedictorian of his class. He then entered the high-school department of the College of the City of New York, and graduated from the College in 1911 at the head of his class. In 1909 he received the Belden Mathematical Prize, in 1909 and 1910 the Pell Medal for the highest rank in all subjects, and in 1910 and 1911 the Kenyon Prize for distinction in pure and applied mathematics. He then entered Columbia University as a graduate student, received the degree of Master of Arts in 1912, and the doctor's degree in 1913. He was a member of Phi Beta Kappa and of Sigma Xi.

While at Columbia, Green became a member of Edward Kasner's seminar. It was here that his interest in projective differential geometry was first aroused, an interest which never flagged and which dominated his whole mathematical career.

In 1913, Dr. Green acted as instructor in mathematics at the College of the City of New York. In 1914 he was appointed to an instructorship at Harvard University, and in

1916 became a member of the faculty. He was a clear, interesting, and inspiring teacher. The department was unanimous in its wish that Green should be promoted to an assistant professorship, and the matter would shortly have been laid before the corporation. But a sudden attack of influenza, followed by pneumonia, cut short this life, so valuable to science and so precious to his friends, before any action could be taken.

Green's musical gifts were hardly inferior to his mathematical talent. He was entirely self taught and was much hampered, in his earlier years, by the inadequacy of the piano which belonged to his parents. Nevertheless, diligent practice on this instrument and a dumb keyboard enabled him to acquire an extensive repertoire and a remarkable technique. His touch was delicate and his musical intuition fine. Music was a form of expression especially well adapted to his emotional and idealistic temperament.

He was short of stature and slender; his features indicated strength and refinement; his expression, always sensitive and often serious, had in it a characteristic undertone of cheerfulness and joy, the joy of a man whose faith in life had not been destroyed, and whose belief in his own powers had not been broken.

Green's first published paper is his Columbia dissertation entitled "Projective differential geometry of triple systems of surfaces," printed privately at Lancaster, Pa., in 1913. The way he wrote and published this thesis is characteristic of his independence and modesty. It had been my task for several years to keep myself informed as to the status and the progress of projective differential geometry; and I had thought that, in the United States at least, no one was likely to be engaged actively in that field without my knowledge. It was with great surprise, then, that I read of a paper with the above title presented to the American Mathematical Society at New York. The title would have aroused my interest at any time, but I had good reason to take more than ordinary interest in this paper just then. I had been engaged for some time in studying this very subject and was nearly ready to prepare my results for publication. I wrote to Green, asking him for some of the details of his work, not suspecting that it had been published already, and to my great astonishment received, a few days later, his printed thesis. This thesis

made it quite unnecessary for me to publish my own work on the subject.

It is unfortunate that this thesis was printed privately; it ought to be reprinted in one of our journals, so as to render its contents generally available. The importance of the subject may be gauged by noting that it may be regarded as containing, among many other things, the theory of triply orthogonal systems, to which subject alone Darboux has devoted a volume of 567 pages.

The theory of triple systems of surfaces, as developed by Green, is based on the theory of invariants and covariants of a completely integrable system of six linear partial differential equations of the second order with one dependent and three independent variables. The subject is far too vast to be treated exhaustively in a thesis of twenty-seven pages; but the fundamental aspects of the theory are there developed and further work in this important field must inevitably set in where Green left off or else duplicate his work.

Closely connected with this thesis are two papers published in 1915 and 1916 in the *American Journal of Mathematics* on "One-parameter families of space curves and conjugate nets on a curved surface." Of course, a continuous one-parameter family of space curves generates a surface, and the properties of this surface are therefore also properties of the one-parameter family of curves. Moreover, these curves determine a second one-parameter family of curves on the surface, such that the two families together form a conjugate system. Finally each of these families determines a congruence, composed of the tangents of the curves of which it is composed. Consequently the theory of a one-parameter family of space curves, the theory of conjugate systems, the theory of surfaces, and the theory of congruences, although distinct, are all intimately connected. In formulating these connections and these distinctions analytically Green exhibits a complete mastery of the situation. The formulas developed by him in his first memoir, the paper of 1915, apply directly to any one-parameter family of curves which are not asymptotic lines on the surface S determined by them. The reason for this exception is easily understood. He uses two independent variables, u and v , one of which, namely v , is constant along one of the curves of the given one-parameter family, while the other changes along such a curve. In every case

then, there presents itself a new one-parameter family of curves on the surface S , the curves $u = \text{const.}$ To simplify the analysis, Green chooses the curves $u = \text{const.}$ to be the curves on S which are conjugate to the given curves $v = \text{const.}$ This can always be done except when the given curves $v = \text{const.}$ are asymptotic curves of S , since asymptotic curves are their own conjugates. Nothing vital is lost by this procedure since the exceptional case of a one-parameter family of space curves, composed of the asymptotic lines of a surface, may be regarded as completely settled by a previously developed theory.

We have purposely discussed this paper, published in 1915, before mentioning an earlier paper of Green's on "One-parameter families of curves in the plane" (*Transactions*, 1914). There was then in existence a closely related theory whose basal features may be described in a very few words. Let $y^{(1)}, y^{(2)}, y^{(3)}$ be the homogeneous coordinates of a point P_v in the plane. A one-parameter family of plane curves may be represented in the form

$$y^{(k)} = f^{(k)}(u, v), \quad (k = 1, 2, 3),$$

where u is the variable which changes when P_v moves along a curve of the given one-parameter family, while v remains constant. Thus v represents the parameter of the family and changes when we pass from one curve of the family to another; the curves of the given family are the curves $v = \text{const.}$ But when formulated in this way, the fact that these functions $f^{(k)}(u, v)$ depend upon both u and v leads one to consider the curves $u = \text{const.}$ as well. The result is a study, not so much of the given one-parameter family itself, as of two one-parameter families, and the net composed of both. This was essentially the situation in which Green found the theory of one-parameter families of plane curves.

In general the second one-parameter family introduced in this way is entirely independent of the first family, and one might say that the properties of the first family are those properties of the net which remain unaltered if the second one-parameter family is changed in all possible ways. A second method of obtaining a theory pertaining entirely to the given one-parameter family of curves would be to show that there exists a second one-parameter family of plane curves completely determined by the first, and then to study

the net composed of the two families. This is precisely what Green did in his theory of one-parameter families of *space* curves by associating, with the given family, the conjugate system. But strange to say, in the theory of *plane* curves, this was a far more difficult matter. The notion of conjugacy breaks down in that case; the idea which might suggest itself of associating with the given curves their orthogonal or certain other isogonal trajectories to constitute a net has to be discarded, since isogonality is a notion not admissible in projective geometry. But Green was able to conquer these difficulties by showing that every one-parameter family of plane curves determines uniquely a second one-parameter family by means of a certain purely projective relation; the theory of the canonical net, as he calls it, which is determined in this way, is then equivalent to the theory of the one-parameter family of plane curves.

In both of the papers just mentioned, Green discusses a question which is of considerable practical importance. It often happens that a theory of invariants has been developed which may be regarded as being general, in so far as it applies to every form of a certain class, but whose practical applicability nevertheless seems doubtful because the formal theory was based, not on the explicitly most general form of its class, but upon a certain canonical form to which every form of the class may be reduced. The obvious way to compute the invariants of such a form would be to reduce it to its canonical form and then to use the formulas for the invariants which the existing theory supplies. But this method cannot always be carried out because the reduction to the canonical form, while possible theoretically, may require operations, like integrations of differential equations, which cannot be carried out in practice.

Such, for instance, was the situation in the projective theory of surfaces at the time when Green wrote these papers. The general formulas were based on the assumption that the surface was referred to its asymptotic lines and, although a method had been given for computing the invariants of a surface when the asymptotic lines were not known explicitly, this method was difficult to apply to special cases. The possibility of using a different method was apparent. Although the actual determination of the asymptotic lines of a surface requires the integration of a quadratic differential

equation of the first order, it is evident that the invariants of the surface must be capable of expression in terms of quantities which define the surface itself rather than its asymptotic lines; they must therefore have expressions which can be made explicit without actually integrating the differential equation of the asymptotic lines. It is Green's merit not only to have appreciated this point, which could not escape him or others, but to have shown that these expressions can be written out in a comparatively simple form. He has actually obtained these general formulas in his paper "On the theory of curved surfaces, and canonical systems in projective differential geometry," *Transactions*, 1915.

The second memoir on "Projective differential geometry of one-parameter families of space curves and conjugate nets on a curved surface" contains a number of interesting and important results. I shall speak in detail merely about one of them. It is well known that the classical metric theory of surfaces may be regarded as the invariant theory of two simultaneous quadratic differential forms which are usually written in the form

$$Edu^2 + 2Fdudv + Gdv^2$$

and

$$Ddu^2 + 2D'dudv + D''dv^2.$$

If $F = 0$, and if the ratio $E : G$ can be expressed as the product of a function of u alone by a function of v alone, the curves $u = \text{const.}$ and $v = \text{const.}$ are said to form an orthogonal isothermal system. If $D' = 0$, the curves $u = \text{const.}$ and $v = \text{const.}$ form a conjugate system. If, besides, the ratio $D : D''$ can be expressed as the product of a function of u alone by a function of v alone, the curves $u = \text{const.}$ and $v = \text{const.}$ are said to be isothermally conjugate, a terminology introduced by Bianchi and justified by the fact that these systems are related to the second differential form in the same way as isothermal orthogonal systems are to the first. Now Bianchi had shown, many years ago, that if a system of curves forms an isothermally conjugate system, this property will be preserved under all projective transformations. It is therefore a projective property. But what is this property? The definition quoted above was purely analytic, and no one had the slightest idea as to what it really meant. Nevertheless this property, whose geometric significance was entirely

hidden, was entering into a continually growing chain of theorems. Could it be admitted under the circumstances that the meaning of any of these theorems was really known? Here then there was a most embarrassing situation. And since the property was known to be of a projective character, the duty of obtaining the desired interpretation clearly devolved upon projective differential geometry. In 1915 I succeeded in finding a simple algebraic relation between certain completely interpreted invariants which was satisfied if and only if the system was isothermally conjugate. But I never felt completely satisfied with my solution of the problem. A great step toward the real solution was taken by Green.

In order to make intelligible Green's condition for isothermal conjugacy, a few preliminary explanations are necessary. Let us remember in the first place that we are dealing with a conjugate system of curves on a surface S . Let P be a point of this surface; let t and t' be the tangents at P of the two curves of the conjugate systems which pass through P ; let a and a' be the asymptotic tangents of the point P . Then t and t' are separated harmonically by a and a' ; this follows immediately from the definition of a conjugate system. Consider the osculating plane at P of each of the two curves of the conjugate system passing through P . They intersect in a line l called the axis of the point P . Thus there is associated with every point P of S a line through P , namely, its axis. The system of ∞^2 lines associated in this way with all of the points of a surface is called the axis congruence. The lines of a congruence can be assembled into a one-parameter family of developables in two ways. Consequently there will pass through P two curves, called axis curves, such that upon each of them the axes of all of its points form developables.

The given conjugate system of curves is composed of two one-parameter families. The tangents of each of these families form a congruence of lines which has S for one sheet of its focal surface. Let P_1 be the point where the tangent t touches the second sheet of the focal surface of the congruence to which it belongs. Let us define P_{-1} similarly on t' : The line P_1P_{-1} is called the ray of the point P . The system of ∞^2 rays which correspond to the ∞^2 points of S is called the ray congruence. The curves on S which correspond to the developables of the ray congruence are its ray curves. Thus we have at a general point of our surface, the two asymptotic

tangents a and a' , the tangents of the given conjugate system t and t' , the pair of axis tangents, and the pair of ray tangents. All of these notions had been formulated by Green's predecessors, although he had been led to consider the axis congruence independently. Now Green defines his *anti-ray tangents*. They are the harmonic conjugates of the ray-curve tangents with respect to t and t' . Moreover there exists a uniquely determined pair of lines which separate both of the pairs a and a' , and t and t' harmonically. These had been considered before, but Green first pointed out their great importance. He calls them the *associate conjugate tangents*:

We have now explained all of the terms in Green's theorem. *A necessary and sufficient condition that a conjugate net on a surface be isothermally conjugate is that for each point of the surface the pair of axis tangents, the pair of anti-ray tangents, and the pair of associate conjugate tangents form pairs of the same involution.* It should be noted however that there is a case, which seems to have escaped Green's attention, which is not settled by this theorem. The exceptional case arises when the axis tangents, and therefore the ray-tangents as well as the anti-ray tangents, separate the tangents of the fundamental conjugate system harmonically. It remains to find a substitute for Green's theorem to cover this case.

I have gone into such details on this point because this theorem, it seems to me, shows better than any other single one of his theorems, the penetrating power of Green's mind, and his fine feeling for mathematical elegance. Perhaps a still simpler geometric condition for isothermal conjugacy will be found; but I am inclined to believe that Green's theorem, in so far as it is applicable, is the final form for this condition.

I now proceed to another paper, also published in 1916, which shows very clearly his great power of generalization. This is his paper on "The linear dependence of functions of several variables and completely integrable systems of homogeneous linear partial differential equations."

It is a familiar fact that $n + 1$ analytic functions $y_1, y_2, \dots, y_n, y_{n+1}$ of a single variable are linearly dependent, if and only if their Wronskian is identically equal to zero. If, at the same time, the Wronskian of $y_1, y_2, \dots, y_n$ is not zero, $y_1, y_2, \dots, y_n$ will be fundamental solutions of a linear homogeneous differential equation

$$(D) \quad \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0,$$

and the Wronskian of $y_1, \dots, y_n$ is capable of a simple expression in terms of the coefficients of this differential equation; in fact it can be expressed in terms of p_1 alone, namely, as a simple exponential function of $\int p_1 dx$, a fact first noted by Abel. A transformation of the form $y = \lambda(x)\bar{y}$ can always be made which will reduce the differential equation (D) to another one of the same form but with a vanishing coefficient for $d^{n-1}\bar{y}/dx^{n-1}$, the so-called canonical form of the differential equation, and this canonical form is unique. It then appears that the remaining coefficients of the canonical form are invariants of (D) under all transformations of the form $y = \lambda(x)\bar{y}$, where λ is an arbitrary function of x .

The analytic discussions of various problems of projective differential geometry had shown clearly in every one of the cases which had actually been investigated that a similar situation exists whenever we consider systems of linearly independent functions of several independent variables, and associate them with completely integrable systems of linear homogeneous partial differential equations. But it required deep insight and a very considerable power of generalization to reveal that the situation thus indicated by the various special theories was actually a universal one. This was accomplished by Green in a most satisfactory way without moreover restricting himself to the case of analytic functions.* It was previously known that the projective differential geometry of a k -spread in a space of any number of dimensions is equivalent to the theory of the invariants and covariants of a certain completely integrable system of linear partial differential equations. Green's theorem, however, enables us to state further that the first stage in the calculation of these invariants, namely the determination of the so-called seminvariants, may always be accomplished explicitly by the same methods which have succeeded in the special cases which have already been studied.

The latest results of Green's geometric studies are closely connected with a relation which he has called the relation R . Green's attention seems to have been called to this relation by the relation between the axis and ray congruences and by a

* In this connection paper No. 9 of the appended list is of special interest.

remarkable fact which presents itself in the theory of surfaces and which I shall recall in a few words. Let us consider a point P on a non-developable surface S and the two asymptotic curves a_1 and a_2 which cross at P . The osculating linear complexes, C_1 and C_2 , of these curves have a linear congruence in common. The directrices, d_1 and d_2 , of this congruence have the property that one of them, d_1 , lies in the plane tangent to S at P but does not pass through P , while the other, d_2 , passes through P but does not lie in the tangent plane. These two lines, d_1 and d_2 , are called the directrices of the first and second kind respectively of the point P . As P moves over the surface S , they generate the corresponding directrix congruences. The curves, called directrix curves, on S which correspond to the developables of these two congruences, are the same for both; i.e., the developables of the two directrix congruences correspond to each other.

The two directrices, d_1 and d_2 , of a surface point are in the relation R , but they furnish only a particular case of this relation. The general relation suggested to Green by this particular case may be explained as follows: Given a surface S and a net composed of two one-parameter families of curves on S , the curves $u = \text{const.}$ and $v = \text{const.}$; let t_1 and t_2 be the tangents of the two curves of the net which meet at P . Given also a line l through every point P of S and not in the corresponding tangent plane. Consider the ruled surface formed by the tangents of the curves $u = \text{const.}$ along a fixed curve $v = \text{const.}$ The plane of l and t_1 will be tangent to this ruled surface at a certain point M_1 of t_1 . In similar fashion we determine a point M_2 on t_2 . The line M_1M_2 , which we shall call l' , is in the tangent plane; it is uniquely determined when the net on S and the line l through P are given. Moreover, unless the given net is conjugate, l is also uniquely determined by l' . Two lines related to one another in this way with respect to a net of curves are said by Green to be in the relation R ; this nomenclature is also extended to the resulting line congruences. In the particular case when the given net is composed of the asymptotic curves of a non-developable surface, the two lines are polar reciprocals with respect to the osculating quadric of the surface point P . I shall hereafter speak of two lines l and l' , which are in the relation R , as Green reciprocals of each other.

I shall mention only a few of the results which Green has

obtained by using this notion. He found an elegant property of isothermal systems which may be formulated as follows: *An orthogonal net on a non-developable surface is an isothermal net, if and only if the developables of the Green reciprocal of the congruence of normals of the surface correspond to a conjugate net on the surface.* He also found the following result which emphasizes the exceptional properties of the directrix congruences. *The two directrix congruences of a surface form the only pair of congruences, whose members are Green reciprocals of each other with respect to the net of asymptotic lines of the surface, and such that the developables of the two congruences correspond to each other.*

The directrix congruences thus have undoubted claims to special attention, and a large number of questions arise when we consider the directrix congruence of the second kind as a kind of projective analogon of the congruence of normals which is so important in the metric theory of surfaces. At one point however, the analogy breaks down. The developables of the normal congruence determine on the surface a net of curves which is not only orthogonal but also conjugate, namely, the lines of curvature. But the directrix curves do not, in general, form a conjugate system. Green has succeeded in defining in purely projective fashion a new pair of congruences, Green reciprocals of each other, one of which may be regarded as being in this sense completely analogous to the normal congruence. He calls it the pseudo-normal congruence and has established its importance in most convincing fashion by a number of applications. One of these is concerned with a fundamental question which has long remained unsolved. Darboux showed, in 1880, that in the neighborhood of a regular point of a surface, the equation of the surface can be expanded in either of the two forms

$$z = xy + \frac{1}{6}(x^3 + y^3) + \frac{1}{24}(Ix^4 + Jy^4) + \dots,$$

$$z = xy + \frac{1}{6}(x^3 + y^3) + \frac{1}{24}xy(Kx^2 + Ly^2) + \dots,$$

where x , y , and z denote non-homogeneous projective coordinates with respect to an appropriately chosen tetrahedron, and where I , J , K , and L , are absolute projective invariants of the surface. But Darboux did not explain the relation of the tetrahedron to the surface for either of these expansions. This question was settled by me about ten years ago, for the

first of these two expansions. Green has now found the corresponding interpretation for the second expansion, and in this interpretation the pseudo-normal plays an important part.

Most of the results of which I have just spoken were announced by Green without proof in the form of brief notes published in the *Proceedings* of the National Academy. The proofs are given in a lengthy memoir, which has appeared in the *Transactions*, April, 1919.

My account of Green's work is far from complete. I have attempted to give only a bare outline of some of his most important work. In the six short years of his mathematical career, from 1913-1919, he enriched geometry with so many new ideas and important results as would suffice to excite our admiration if they had been spread over all of a normal life time. In his death we have suffered a heavy loss, but his life and work will continue to be, for many of us, an everlasting source of strength and inspiration.

CHICAGO,
March 29, 1919.

LIST OF GREEN'S PUBLISHED PAPERS.

1913.

1. Projective differential geometry of triple systems of surfaces. Columbia dissertation. Published privately. New Era Press, Lancaster, Pa.

1914.

2. One-parameter families of curves in the plane. *Transactions of the American Mathematical Society*, vol. 15, No. 3, pp. 277-290, July.

1915.

3. On the theory of curved surfaces and canonical systems in projective differential geometry. *Transactions of the American Mathematical Society*, vol. 16, No. 1, pp. 1-12, January.
4. Projective differential geometry of one-parameter families of space curves and conjugate nets on a curved surface. (First memoir.) *American Journal of Mathematics*, vol. 37, No. 3, pp. 215-246, July.
5. On isothermally conjugate nets of space curves. *Proceedings of the National Academy of Sciences*, vol. 1, No. 10, pp. 516-521, October.

1916.

6. On the linear dependence of functions of several variables and certain completely integrable systems of partial differential equations. *Proceedings of the National Academy of Sciences*, vol. 2, No. 4, pp. 209-214, April.
7. Projective differential geometry of one-parameter families of space curves and conjugate nets on a curved surface. (Second memoir.) *American Journal of Mathematics*, vol. 38, No. 3, pp. 287-324, July.

8. The linear dependence of functions of several variables and certain completely integrable systems of partial differential equations. *Transactions of the American Mathematical Society*, vol. 17, No. 4, pp. 483-516, October.
9. On the linear dependence of functions of one variable. *Bulletin of the American Mathematical Society*, vol. 23, No. 3, pp. 118-122, December.

1917.

10. On the general theory of curved surfaces and rectilinear congruences. *Proceedings of the National Academy of Sciences*, vol. 3, No. 10, pp. 587-592, October.
11. Some geometric characterizations of isothermal nets on a curved surface. *Transactions of the American Mathematical Society*, vol. 18, No. 4, pp. 480-488, October.

1918.

12. Note on conjugate nets with equal point invariants. *Bulletin of the American Mathematical Society*, vol. 24, No. 5, pp. 221-225, February.
13. The intersections of a straight line and hyperquadric. *Annals of Mathematics*, ser. 2, vol. 19, No. 3, pp. 207-209, March.
14. Plane nets with equal invariants. *Annals of Mathematics*, ser. 2, vol. 19, No. 4, pp. 246-250, June.
15. On certain projective generalizations of metric theorems and the curves of Darboux and Segre. *Proceedings of the National Academy of Sciences*, vol. 4, No. 11, pp. 346-349, November.

1919.

16. Memoir on the general theory of surfaces and rectilinear congruences. *Transactions of the American Mathematical Society*, vol. 20, No. 2, pp. 79-153, April.

REDUCTION OF THE ELLIPTIC ELEMENT TO THE WEIERSTRASS FORM.

BY PROFESSOR F. H. SAFFORD.

(Read before the American Mathematical Society April 26, 1919.)

IN Enneper's *Elliptische Functionen*, page 27, may be found a method due to Weierstrass of reducing the general elliptic element to the Weierstrass form. Briefly, from

$$(1) \quad x = x_0 + \frac{\sqrt{R(x_0)} \sqrt{(4s^3 - g_2s - g_3)} + \frac{1}{2}R'(x_0)[s - \frac{1}{24}R''(x_0)] + \frac{1}{24}R(x_0)R'''(x_0)}{2[s - \frac{1}{24}R''(x_0)]^2 - \frac{1}{2}A \cdot R(x_0)}$$

comes

$$(2) \quad \frac{dx}{\sqrt{R(x)}} = \frac{-ds}{\sqrt{(4s^3 - g_2s - g_3)}} = du.$$

Enneper adds: "Diese Resultate wurden zuerst veröffentlicht von W. Biermann: 'Problemata quædam mechanica functionum ellipticarum ope soluta,' Diss. Berlin, 1864," and the latter states that they were given in Weierstrass's lectures.

Necessary additional formulas are

$$\begin{aligned} R(x) &= Ax^4 + 4Bx^3 + 6Cx^2 + 4B'x + A' \\ &= r_0 + r_1(x - x_0) + r_2(x - x_0)^2 + r_3(x - x_0)^3 \\ &\quad + r_4(x - x_0)^4, \end{aligned}$$

$$\begin{aligned} r_0 &= R(x_0), & r_1 &= R'(x_0), & r_2 &= R''(x_0)/2, \\ r_3 &= R'''(x_0)/6, & r_4 &= R''''(x_0)/24, \end{aligned}$$

$$\begin{aligned} (3) \quad g_2 &= AA' + 3C^2 - 4BB' = r_0r_4 - \frac{1}{4}r_1r_3 + \frac{1}{2}r_2^2 \\ &= 4\epsilon_1\epsilon_2\epsilon_3, \\ g_3 &= ACA' + 2BCB' - AB'^2 - A'B^2 - C^3 \\ &= \frac{1}{8}r_0r_2r_4 + \frac{1}{4}r_1r_2r_3 - \frac{1}{16}r_0r_3^2 - \frac{1}{16}r_4r_1^2 - \frac{1}{2}\frac{1}{16}r_2^3 \\ &= -4(\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1), \end{aligned}$$

$$s = \wp(u + c), \quad (x_0 \text{ and } c \text{ are arbitrary constants}).$$

Enneper later uses (1) in obtaining the addition formula for $\wp(u + v)$, and Greenhill also quotes (1), adding a brief note.

Haentzschel, in *Reduction der Potentialgleichung*, discussing surfaces of revolution in a potential problem, obtains, following Wangerin, this differential equation

$$\begin{aligned} (4) \quad F'^2(u) &= AF^4(u) + 4BF^3(u) + 6CF^2(u) \\ &\quad + 4B'F(u) + A' = R(F), \end{aligned}$$

for which he gives a solution

$$(5) \quad x = F(u) = \alpha + \frac{R'(\alpha)/4}{s - R''(\alpha)/24},$$

which is obtainable from (1) by using for x_0 the value α , any root of $R(x) = 0$. Haentzschel adds “. . . so könnte nach Herrn Weierstrass der Ausdruck (5) doch noch nicht als die allgemeinste Form des Integrals von (4) gelten, sondern als solchen leitet Herr Weierstrass ab,” and then gives (1) above. Haentzschel uses the simpler solution in obtaining various families of orthogonal curves according to the methods of conjugate functions, but merely gives an outline for the case of the (apparently) more general solution.

The writer has given discussions and extensions of Haentzschel's results in the *Archiv der Mathematik und Physik*. In addition to obtaining the equations of the curves corresponding to the more general solution, he has shown that these equations are factorable and include the curves from the simpler case. These results, omitting computations in general, have appeared in this BULLETIN, June, 1899, March, 1912, November, 1917.

The object of the present paper is to show that solution (1) arises solely from a particular choice of constants in the simpler form (5). But (5) is itself of considerable interest because it may be used to obtain the equation referred to by Durège as “Die Landensche Substitution.” If $R(x)$ be replaced by Legendre's form, (5) gives

$$(6) \quad x = \operatorname{sn}(u\sqrt{e}) = \sqrt{(e)(s - \epsilon_1)/(s - \epsilon_2)(s - \epsilon_3)},$$

while the commonly quoted relation between sn and $\wp$ is

$$(7) \quad \operatorname{sn}(u\sqrt{e}) = \sqrt{e}/\sqrt{(t - a_3)},$$

in which t and a_3 correspond respectively to s and ϵ_3 ; in both formulas e is essentially A' used above. From (6) and (7) the substitution may be derived. Proceeding to the more important derivations, the fundamental equation (1) is solved for s , giving

$$(8) \quad \begin{aligned} s &= \wp(u + c) \\ &= \frac{\sqrt{r_0}\sqrt{R(x)} + r_0 + r_1(x - x_0)/2 + r_2(x - x_0)^2/6}{2(x - x_0)^2}. \end{aligned}$$

Next let x_0 have the value α , where α is a root of $R(x) = 0$, so that the corresponding form of (8) is

$$\begin{aligned}
 s_1 &= \wp(u + c_1) \\
 (9) \quad &= \frac{t_0 + t_1(x - \alpha)/2 + t_2(x - \alpha)^2/6}{2(x - \alpha)^2} \\
 &= \frac{t_1/4 + t_2(x - \alpha)/12}{x - \alpha}.
 \end{aligned}$$

It should be noted that in (9)

$$(10) \quad t_0 = R(\alpha) = 0, \quad t_1 = R'(\alpha), \quad t_2 = R''(\alpha)/2.$$

As c and c_1 are merely constants of integration in solutions of the same differential equation, it is essential to show what relation between them will make the two solutions identical. By the aid of the usual formula

$$\begin{aligned}
 &[\wp(u) + \wp(v)][4\wp(u)\wp(v) - g_2] \\
 (11) \quad \wp(u \pm v) &= \frac{-2g_3 \mp 2\wp'(u)\wp'(v)}{4(\wp(u) - \wp(v))^2}
 \end{aligned}$$

from (8) and (9) follows

$$(12) \quad c - c_1 = \wp^{-1}(s) - \wp^{-1}(s_1)$$

or

$$(13) \quad s_2 = \wp(c - c_1) = \frac{(s + s_1)(4ss_1 - g_2) - 2g_3 - 2s's_1'}{4(s - s_1)^2}.$$

The direct reduction of the second member of (13) would be extremely long, but it can be readily carried far enough by computing the coefficients of certain powers of x to show that

$$(14) \quad s_2 = \wp(c - c_1) = \frac{r_0 + r_1(\alpha - x_0)/2 + r_2(\alpha - x_0)^2/6}{2(\alpha - x_0)^2},$$

which is the desired relation mentioned above. It is important to notice that s_2 is obtainable from s by replacing x by α , while s_1 was similarly obtained by replacing x_0 by α . In both cases the radicals in s vanish. Though not necessary, a verification of the preceding result, (14), may be effected without difficulty with the aid of (11) and (12) by computing

$$(15) \quad s = \wp[\wp^{-1}(s_1) + \wp^{-1}(s_2)] = \wp(u + c).$$

Thus (8) or its equivalent form, (1), the fundamental formula due to Weierstrass, is obtainable directly from the simple solution (5) by a suitable choice of the constant c as given by (14).

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA.

A NOTE ON "CONTINUOUS MATHEMATICAL INDUCTION."

BY DR. YUEN REN CHAO.

(Read before the San Francisco Section of the American Mathematical Society April 5, 1919.)

1. *Special case.*—Let the function $f(x)$ be defined in some interval of a real variable x .

Hyp. 1. Let there be a point a in the interval such that $f(a) = 0$.

Hyp. 2. Let there be a constant Δ for the interval, such that $f(x) = 0$ implies $f(x + \delta) = 0$, whenever $0 < \delta \leq \Delta$.

Then for any b in the interval, where $b > a$, $f(b) = 0$.

Proof.—I. If $b - a \leq \Delta$, then by Hyp. 2 the conclusion follows.

II. If $b - a > \Delta$, then first apply Archimedes' postulate, that is, there will be an integer n and a fraction θ ($0 \leq \theta \leq 1$) such that

$$b - a = (n + \theta)\Delta, \quad \text{or} \quad b = (a + \theta\Delta) + n\Delta.$$

Next, apply ordinary mathematical induction, thus: By Hyp. 1 and 2, since $\theta\Delta < \Delta$,

$$\therefore f(a + \theta\Delta) = 0.$$

Therefore, by 2, again,

$$(1) \quad f[(a + \theta\Delta) + 1 \cdot \Delta] = 0.$$

By 2, if $f[(a + \theta\Delta) + m \cdot \Delta] = 0$, then

$$(2) \quad f[(a + \theta\Delta) + (m + 1)\Delta] = 0.$$

Hence, combining (1) and (2),

$$f(a + \theta\Delta + n\Delta) = 0,$$

that is,

$$f(b) = 0.$$

2. *General case.*—Let $\varphi(x)$ be any propositional function, defined in some interval of a real variable x .

Hyp. 1. Let there be a point a in the interval such that $\varphi(a)$ is true.

Hyp. 2. Let there be a constant Δ for the interval such that $\varphi(x)$ implies $\varphi(x \pm \delta)$, whenever $0 < \delta \leq \Delta$.

Then for any b in the interval such that $b \geq a$, respectively, $\varphi(b)$ is true.

The proof will be the same as for the special case, except for obvious changes of wording or sign.

Remarks.—The theorem rests essentially on Archimedes' postulate and on ordinary mathematical induction, but it is not a generalization of the latter, in the sense of including it as a special case. It is not a theorem in mathematical logic, since it is concerned with a real variable x . But it is more general than ordinary theorems dealing with equalities, in that $\varphi(x)$ may be a statement about continuity, convergence, integrability, etc., that cannot be put in the simple form of $f(x) = 0$.

The theorem is a mathematical formulation of the familiar argument from "the thin end of the wedge," or again, the argument from "the camel's nose":

Hyp. 1. Let it be granted that the drinking of half a glass of beer be allowable.

Hyp. 2. If any quantity, x , of beer is allowable, there is no reason why $x + \delta$ is not allowable, so long as δ does not exceed an imperceptible amount Δ .

Therefore any quantity is allowable.

Like all mathematical theorems, the conclusion is no surer than its hypothesis. In this case, if the argument fails, it is usually because a *constant* Δ required in the second hypothesis does not exist. Take the very wedge itself. If a wedge is driven with a constant force between two sides which are pushed together by elastic forces, it will be stopped when balanced by the component of the increasing resistance. In this case the Δ within which δ may increase for $\varphi(x + \delta)$ to continue to hold will not be "uniform for the interval," so to speak, but will become smaller and smaller as x approaches the dangerous point, beyond which the conclusion ceases to be true.

UNIVERSITY OF CALIFORNIA.

ON THE NUMBER OF REPRESENTATIONS OF $2n$
AS A SUM OF $2r$ SQUARES.

BY PROFESSOR E. T. BELL.

(Read before the San Francisco Section of the American Mathematical Society April 5, 1919.)

1. OWING possibly to its connection with X-ray analyses of crystal structure, interest in the problem of representing an integer as a sum of integral squares has recently revived. We shall first summarize briefly so much of what is known of the problem as will put the formulas established below in their proper light. Let $N(n, 2r)$ denote the total number of decompositions of n into a sum of $2r$ squares. Then, for $r = 1, 2, 3, 4$, the complete results concerning $N(n, 2r)$ are implicit in sections 40, 41, 42, 65 of the *Fundamenta Nova*. Jacobi, however, left the explicit statement of all but one of his results to others. When $r \geq 4$, $N(n, 2r)$ is expressible in terms of the divisors of n alone. By arithmetical methods, independently of elliptic functions, Eisenstein* proved some of Jacobi's results, showed how the rest might be obtained from his own theorems, and proved that, for $r > 4$ and n general, $N(n, 2r)$ can not be expressed in terms of the divisors of n alone. Letting $\xi_s'(n)$ denote the excess of the sum of the s th powers of all those divisors of n that are of the form $4k + 3$ over the like sum for all divisors of the form $4k + 1$, and $\zeta_s(n)$ the sum of the s th powers of all the divisors, Eisenstein stated a notable exception to his general theorem; showing that at least once, when n is suitably chosen, $N(n, 2r)$, for $r > 4$, may be expressed in terms of $\xi_s'(n)$, or $\zeta_s(n)$. E.g., $N(4k + 3, 10) = 12\xi_4'(4k + 3)$; or what may be shown is ultimately the same thing:† $N(8k + 6, 10) = 204\xi_4'(4k + 3)$. Liouville derived a similar result for $N(2m, 12)$ in terms of $\zeta_6(2m)$. He used for this purpose certain remarkable formulas‡ which, however, he did not prove, and which it is the

* *Crelle*, vol. 35 (1847), p. 135.

† Either result follows from the other on applying a transformation of the second order to the theta equivalent of the appropriate Liouville formula of the kind in § 2.

‡ *Jour. des Math.* (2), vol. 6 (1861); two papers, p. 233, 369 et seq. The formulas for proper representations may be proved similarly to those in this paper.

object of this paper to establish; he also showed how the discovery of such results as Eisenstein's and his own can be made to depend upon the formulas mentioned. In 1907 G. Humbert* and K. Petr† independently proved Eisenstein's 10-square, and Liouville's 12-square results very simply by elliptic functions. At the same time J. W. L. Glaisher‡ published complete results for $N(n, 2r)$, $r = 1$ to 9, inventing the necessary functions for the cases $r = 5$ to 9. He remarked§ that the form of his results seems to indicate the non-existence for $r = 7, 8, 9$ of theorems similar to Eisenstein's or Liouville's for $r = 5, 6$. Recently L. J. Mordell|| has found and developed close connections between Glaisher's theorems and the elliptic modular invariants. Liouville's formulas seem to have been overlooked by later writers. They offer a direct method of attack upon the question of completely expressing $N(2n, 2r)$ in terms of the divisors of n alone; that is, of finding for what forms of n, r this is possible. The four formulas of Liouville are only the first cases of an infinite number of similar results which may be found as below, using higher powers than the first and second, or products, etc., of the elliptic series; and like results exist for $N(m, 2r)$ where m is odd, Eisenstein's theorem being a consequence of one of these. As Liouville remarks, there is an extensive theory in connection with such formulas. Here we shall merely prove his four, and show how the required coefficients may be found.

2. For m odd, Liouville's formulas are

$$\xi_s(m) \equiv (-1)^{(m+1)/2} \xi'_s(m);$$

$$(1) \quad \xi_{2r-1}(m) = \sum_{t=0}^{r-1} A_t N(2m, 4r, 4t+2), \quad (r > 0);$$

$$(2) \quad \xi_{2r}(m) = \sum_{t=0}^r B_t N(2m, 4r+2, 4t+2), \quad (r \geq 0);$$

and for $n = 2^{a+2}m$, m odd, $a \geq 0$

$$(3) \quad 2^{(2r+1)a} \xi_{2r+1}(m) = \sum_{t=0}^{r-1} \alpha_t N(n, 4r+4, 4t+4), \quad (r > 0);$$

* *Paris C. R.*, vol. 144, p. 874.

† *Archiv der Math.*, 1907, p. 83.

‡ (1) *Q. J. M.*, vol. 38. The results are summarized in (2): *Proc. L. M. S.* (2), vol. 5 (1907), pp. 479-90.

§ *Loc. cit.* (2), p. 487, § 13.

|| *Q. J. M.*, vol. 48 (1917-18), Nos. 189, 190.

$$(4) \quad 2^{2ar} \xi_{2r}(m) = \sum_{t=0}^{r-1} \beta_t N(n, 4r+2, 4t+4), \quad (r > 0);$$

where $N(n, r, s)$ = the number of representations of n as a sum of r squares, of which the first s are odd with roots > 0 , and the last $r-s$ even with roots ≤ 0 . The coefficients A , B , α , β depend upon r , but not upon m or n . Liouville states that for all values of r

$$A_0 = B_0 = \alpha_0 = \beta_0 = 1;$$

$$C_{r-1} = 16^{r-1}, \quad C_{r-t-1} = 16^{r-2-t-1} C_t,$$

where C denotes either A or α ; $B_r > 0$ when $r > 0$; and that recurring formulas exist whereby the successive coefficients may be calculated from the first. These assertions will be verified automatically in the proofs. We shall derive (1) from a comparison of the power series and Fourier developments of $\text{sn}u$; and (2), (3), (4) in a like manner from $\text{cn}u$, sn^2u , $\text{dn}u$ respectively.

3. The necessary series are, with $0! = 1$:

$$(5) \quad \begin{aligned} \text{sn}u &= \frac{2\pi}{kK} \sum \frac{q^{m/2}}{1 - q^m} \sin \frac{m\pi u}{2K} \\ &= \sum_{r=1}^{\infty} S_{2r-1}(k^2) (-1)^{r-1} \frac{u^{2r-1}}{(2r-1)!}; \end{aligned}$$

$$(6) \quad \text{cn}u = \frac{2\pi}{kK} \sum \frac{q^{m/2}}{1 + q^m} \cos \frac{m\pi u}{2K} = \sum_{r=0}^{\infty} C_{2r}(k^2) (-1)^r \frac{u^{2r}}{(2r)!};$$

$$(7) \quad \begin{aligned} \text{dn}u &= \frac{2\pi}{K} \left[\frac{1}{4} + \sum \frac{q^n}{1 + q^{2n}} \cos \frac{n\pi u}{K} \right] \\ &= \sum_{r=0}^{\infty} D_{2r}(k^2) (-1)^r \frac{u^{2r}}{(2r)!}; \end{aligned}$$

$$(8) \quad \begin{aligned} \text{sn}^2u &= 2 \left(\frac{\pi}{kK} \right)^2 \left[\sum \frac{q^m}{(1 - q^m)^2} - \sum \frac{nq^n}{1 - q^{2n}} \cos \frac{n\pi u}{K} \right] \\ &= \sum_{r=1}^{\infty} S_{2r}'(k^2) (-1)^{r-1} \frac{u^{2r}}{(2r)!}; \end{aligned}$$

the first sums being with respect to $n = 1, 2, 3, 4, \dots$, and $m = 1, 3, 5, 7, \dots$. The S , C , D , S' are polynomials in k^2 with positive integral coefficients, and their known forms are

$$(9) \quad S_{2r-1}(k^2) = \sum_{t=0}^{r-1} s_t k^{2t}; \quad C_{2r}(k^2) = \sum_{t=0}^{r-1} c_t k^{2t};$$

$$S_{2r}'(k^2) = \sum_{t=0}^{r-1} s_t' k^{2t}; \quad D_{2r}(k^2) = k^{2r} C_{2r}(1/k^2),$$

the last from the relation $\text{cn}(ku, 1/k) = \text{dn}u$. Also, for all values of r we have *

$$(10) \quad s_0 = c_0 = 1; \quad s_0' = s_{r-1}' = 2^{2r-1}; \quad s_{r-t-1} = s_t;$$

$$s_{r-t-1}' = s_t'; \quad c_{r-1} = 2^{2r-2}.$$

Henceforth, unless otherwise indicated, n represents an arbitrary integer > 0 , m an arbitrary odd integer > 0 , and summations are with respect to all m or all n . We shall need the following constants:

$$(11) \quad \Sigma \frac{q^{m/2} m^{2r-1}}{1 - q^m} = \Sigma q^{m/2} \zeta_{2r-1}(m); \quad \Sigma \frac{q^{m/2} m^{2r}}{1 + q^m} = \Sigma q^{m/2} \xi_{2r}(m),$$

the right members coming directly from the definitions of ζ , ξ' in § 1, on expanding each term on the left and rearranging all in ascending powers of q . Writing for the moment

$$(12) \quad n = 2^a m, \quad \xi_r''(n) = 2^{ar} \xi_r(m), \quad \zeta_r'(n) = 2^{ar} \zeta_r(m),$$

we have also

$$(13) \quad \Sigma \frac{q^n n^{2r}}{1 + q^{2n}} = \Sigma q^n \xi_{2r}''(n); \quad \Sigma \frac{q^n n^{2r+1}}{1 - q^{2n}} = \Sigma q^n \zeta_{2r+1}'(n).$$

The necessary theta constants are, $\vartheta_a \equiv \vartheta_a(q)$:

$$(14) \quad \sqrt{\frac{2K}{\pi}} = \vartheta_3 = \sum_{n=-\infty}^{+\infty} q^{n^2}; \quad k = \frac{\vartheta_2^2}{\vartheta_3^2}; \quad \vartheta_2 = 2 \Sigma q^{m^2/4};$$

whence, for b, c integers ≥ 0 , we have, obviously

$$(15) \quad \vartheta_2^b(q^4) \vartheta_3^c(q^4) = \Sigma q^n [2^b N(n, b + c, b)].$$

For passing from representations to compositions, the following is useful:

$$(16) \quad \vartheta_2^b(q^4) \vartheta_3^c(q^4) = \Sigma q^n [N'(n, b + c, b)],$$

* The third of the relations (10) is from $\text{sn}(kx, 1/k) = k \text{sn}(x, k)$; and the fourth follows from this by actually forming the product $\text{sn}u \times \text{sn}u$, and noting that the coefficient is necessarily a reciprocal polynomial in k^2 . It is important for the verification of Liouville's α_i coefficients to observe that $s_0' = s_{r-1}' = 2^{2r-1}$, which may be seen at once from $S_{2r}'(k^2)$, whose absolute term is $\frac{1}{2}[(1+1)^{2r} - (1-1)^{2r}]$.

where $N'(n, p, q)$ is the number of representations of n as a sum of p squares, the first q of which are odd with roots ≥ 0 , and the last $p - q$ even with roots ≤ 0 .

4. Equating coefficients of $(-1)^{r-1}u^{2r-1}/(2r-1)!$ in (5), we find, on using (11),

$$(17) \quad \frac{4}{k} \left(\frac{\pi}{2K} \right)^{2r} \Sigma q^{m/2} \zeta_{2r-1}(m) = S_{2r-1}(k^2) \equiv \sum_{i=0}^{r-1} s_i k^{2i},$$

which, on substituting for k, K their values from (14), becomes, after some obvious reductions,

$$(18) \quad 4 \Sigma q^{m/2} \zeta_{2r-1}(m) = \sum_{i=0}^{r-1} s_i \vartheta_2^{4i+2} \vartheta_3^{4r-4i-2}.$$

From (6), in the same way,

$$(19) \quad 4 \Sigma q^{m/2} \xi_{2r}(m) = \sum_{i=0}^{r-1} c_i \vartheta_2^{4i+2} \vartheta_3^{4r-4i}.$$

Changing q into q^4 in (18), (19), and applying (15), we get

$$(20) \quad \zeta_{2r-1}(m) = \sum_{i=0}^{r-1} 2^{4i} s_i N(2m, 4r, 4i+2);$$

$$(21) \quad \xi_{2r}(m) = \sum_{i=0}^{r-1} 2^{4i} c_i N(2m, 4r+2, 4i+2);$$

by equating the coefficients of q^{2m} in the respective series. To verify Liouville's statements about the coefficients A_i, B_i of § 2, we note that (20), (21) become respectively (1), (2) on putting $A_i = 2^{4i} s_i, B_i = 2^{4i} c_i$; hence by (10) the verification is complete. The calculation for (8) is similar to the foregoing. We find first, on using (13),

$$\frac{2}{k^2} \left(\frac{\pi}{K} \right)^{2r+2} \Sigma q^n \zeta'_{2r+1}(n) = \sum_{i=0}^{r-1} s'_i k^{2i};$$

which reduces to

$$(22) \quad 2^{2r+3} \Sigma q^n \zeta'_{2r+1}(n) = \sum_{i=0}^{r-1} s'_i \vartheta_2^{4i+4} \vartheta_3^{4r-4i};$$

and this, on expressing ζ' in terms of ζ , by (12), substituting q^4 for q , and using (16), becomes, for $n = 2^a m$,

$$(23) \quad 2^{a(2r+1)} \zeta_{2r+1}(m) = \sum_{i=0}^{r-1} 2^{4i-2a+1} s'_i N(4n, 4r+4, 4i+4).$$

Corresponding to (22), (7) gives, on substituting for D_{2r} its value in terms of C_{2r} from (9),

$$(24) \quad 2 \left(\frac{\pi}{K} \right)^{2r+1} \Sigma q^n \xi_{2r}''(n) = k^{2r} \sum_{i=0}^{r-1} c_i k^{-2i};$$

whence

$$(25) \quad 2^{2r+2} \Sigma q^n \xi_{2r}''(n) = \sum_{i=0}^{r-1} c_i \partial_2^{4r-4i} \partial_3^{4i+2};$$

$$(26) \quad 2^{2ar} \xi_{2r}(m) = \sum_{i=0}^{r-1} c_i 2^{2r-4i-2} N(4n, 4r+2, 4r-4i),$$

the last by changing q into q^4 in (25) and equating coefficients of q^{4n} ; $n = 2^a m$, $a \geq 0$. This may clearly be rewritten

$$(27) \quad 2^{2ar} \xi_{2r}(m) = \sum_{i=0}^{r-1} 2^{4i-2r+2} c_{r-i-1} N(4n, 4r+2, 4i+4).$$

Again, on writing $\alpha_i = 2^{4i-2r+1} s'_i$, $\beta_i = 2^{4i-2r+2} c_{r-i-1}$, we see by (10) that these coefficients have the properties announced by Liouville, and that (23), (27) are identical with (3), (4). The r -limitations $r \geq 0$, etc., in (1) to (4) are obviously met.

5. The coefficients s , s' of (20), (23) may be calculated by recurrence, as shown by Gudermann.* The coefficients of (21), (27) are more easily found by Hermite's method.† But whichever method is used, the computation involves great labor if $r > 20$; and neither gives the c , s , s' as explicit functions of r , t . It would be desirable, for the consideration of several questions related to compositions as sums of squares, to have these functions of r , t explicitly. Thus, e.g., were they known, it would be possible, following Liouville's indications,‡ to state necessary and sufficient conditions that

* *Crelle*, vol. 19 (1839), pp. 79-83. Some of the coefficients in Gudermann's § 115 seem to be incorrect, but the essential part of his method on p. 79 is unaffected by this. Another recurrence for the s -coefficients is given by $\text{sn}'u = \text{cn}u \text{dn}u$, and this one enables us to apply Hermite's method to the calculation of the s , s' as well as the c .

† "Rémarques sur le développement de $\cos amx$;" *J. des Math.* (2), vol. 9 (1869), p. 289; *Paris C. R.*, vol. 77 (1863), p. 613. Apparently the quantities first denoted by $A_0, A_1, \dots, A_n$ in Hermite's paper should each be multiplied by $(-1)^{n+1}$. In the rest of the paper they are taken with this meaning, so that the final results appear as intended. The misprint is of no importance for Hermite's purpose, but it is for ours; since otherwise the coefficients would not all be positive.

‡ *J. des Math.* (2), vol. 6, pp. 234-5. If (28)-(31) of the present paper are used in this connection, the powers of 2 do not appear as factors (as in Liouville's illustration), but only the binomial coefficients.

$N(2n, 2r)$ shall be a ξ , ζ -function of the divisors of n alone, for general r and n . The coefficients s , s' , c , and hence A , B , α , β are clearly all integers > 0 ; but beyond this, and the few facts concerning them which were observed by Liouville, little if anything of arithmetical importance is known about them. The comparative largeness of their prime factors is significant.

6. It will be worth while, for its bearing on the question of composition, to write down the formulas which correspond to (1)–(4), when the restriction that the odd squares shall have positive roots is removed. They are, from (15), (16), (20), (21), (23), (27), as follows:

$$(28) \quad 4\zeta_{2r-1}(m) = \sum_{i=0}^{r-1} s_i N'(2m, 4r, 4i+2),$$

$$(29) \quad 4\xi_{2r}(m) = \sum_{i=0}^{r-1} c_i N'(2m, 4r+2, 4i+2),$$

$$(30) \quad 2^{a(2r+1)+2b+s} \zeta_{2r-1}(m) = \sum_{i=0}^{r-1} s'_i N'(n, 4r+4, 4i+4),$$

$$(31) \quad 2^{2(a+r+1)} \xi_{2r}(m) = \sum_{i=0}^{r-1} c_{r-i-1} N'(n, 4r+2, 4i+4),$$

where m , n , r are as in § 2.

7. For $N(n, 2r)$ as defined in § 1, we have

$$N(n, 2r) = N'(n, 2r, 0) + \binom{2r}{1} N'(n, 2r, 1) \\ + \binom{2r}{2} N'(n, 2r, 2) + \cdots + N'(n, 2r, 2r),$$

where $\binom{2r}{s} = a!/b!(a-b)!$. Hence, it is easily seen from (28), a sufficient condition that $N(2m, 4r)$ shall be expressible in terms of the divisors of $2m$ alone is that the ratio $s_i : \binom{4r}{4i+2}$ shall have the same value for $i = 0, 1, 2, \dots, r-1$. This condition is satisfied for $r = 3$. When $r = 4, 5, 6$, it is not. Or again, if some but not all of $s_1, s_2, \dots, s_{r-2}$ are divisible by one prime $> 4r$, then clearly the condition is not satisfied. This criterion rejects $r = 5$, the prime factor 307 occurring in two out of the possible three terms; or this case is rejected by 83, which is a factor of only the middle term. Similar conclusions may be read from (29), (30), (31).

THE UNIVERSITY OF WASHINGTON.

SOME FUNCTIONAL EQUATIONS IN THE
THEORY OF RELATIVITY.

BY PROFESSOR ARTHUR C. LUNN.

(Read before the American Mathematical Society April 26, 1919.)

IN transcribing the process of light-signalling which leads to his kinematic equations of transformation, Einstein* obtains relations, connecting the space-time coordinates in two systems of reference, in the form of functional equations. He solves these by passing to partial differential equations, and restricts the solutions to linearity by a rather obscure appeal to the homogeneity of time and space. In view of the fundamental character that his theory has proved to possess it is natural to ask more closely what is implied in the term homogeneity, which occurs in various senses in physical science, and whether it is necessary to assume differentiability. These and some related questions bearing on the fundamentals of that theory suggested the following analysis, which aims at a more complete transcription by means of functional equations and their solution without appeal to differentiation.

The general setting of the problem is found in the supposition that for the set of space-time points there are various systems of reference, corresponding to observers and carriers of coordinate frames in uniform translatory motion with respect to each other; these systems being, under conditions to be described, all on a parity with each other in the sense to be defined as the meaning of relativity. It is understood that every space-time point is identifiable by each system in terms of a unique quadruple of coordinates, and conversely that each such quadruple that occurs at all identifies a unique space-time point. For a physical theory some limitation on the range of the coordinates might perhaps seem appropriate, and for the most part even if such a limitation of any natural kind were imposed the results would probably be unchanged if interpreted for corresponding regions. But in order to avoid the complication of boundary conditions that would accompany such a limitation it will be supposed that every

* Einstein, *Drude Annalen*, vol. 17, pp. 891-921 (1905).

point of euclidean space of three dimensions exists at each instant for every observer, that no other points ever exist, and that the space-time manifold is in a certain impartial sense continuous. For analytic purposes this is taken to mean that identification of the same space-time points by any two observers yields a unique one-to-one continuous correspondence of their quadruples of coordinate values, existing for the entire infinite real range. In other words the coordinates (τ, ξ, η, ζ) in any system Σ are single-valued continuous functions of the coordinates (t, x, y, z) in any other system S , for all real values of the latter; these functions having then in some definite sense, though perhaps not algebraically, single-valued inverses. The problem is to determine the form of the functions by the imposition of conditions suitable to the notion of relativity as understood in the Einstein theory. For each system by itself the full kinematics of euclidean space and optical time is taken for granted.

Einstein's postulate of the constancy of the velocity of light has been much discussed, yet seems to invite still more detailed analysis than it has hitherto received. It clearly involves along with its physical content a considerable element of convention, with respect to orientation of axes and choice of units for instance. The essential physical feature seems to be that there is supposed to exist in every system by itself a cartesian scheme of measurement for space and time such that as interpreted in terms of that scheme space is optically isotropic; that is, light travels in straight lines with a velocity constant for each line and the same for all. This involves of course that much-discussed independence of the motion of the source of the light. For Einstein's plan of testing clocks by means of to-and-fro light signals, however, the rectilinearity of the light path is not directly made use of. It is in effect assumed that, starting at any instant, a light-signal can be sent from any point to any other, and that the distance is proportional to the time of transit. The factor of proportionality, the velocity of light, is assumed to be the same for all pairs of points, in any single system, and for convenience is taken as having the same numerical value c in every system by means of a single admissible restriction on the units of length and time in each. For the present purpose it will be assumed that in every system the corresponding scheme of measurement is of the kind described. The nature of the conditions

to be imposed is such that there remain arbitrary in each system the choice of origin, euclidean transformation of space-axes, and one further condition on the units.

That the system Σ as observed by S has a uniform translatory rigid motion means: that all points which in Σ appear fixed have in S a common velocity, of constant direction and of constant magnitude v , and conversely that any set of points having in S this particular velocity appear in Σ fixed. For consideration of a particular pair of systems let the direction of this velocity specify the direction of the x -axis in S . The light signals are to be thought of as passing between points fixed in Σ and described in S . A signal is longitudinal when these points are at one instant and therefore always on a line in S parallel to the x -axis; it is lateral when the points lie on a moving line perpendicular to the x -axis.

A forward longitudinal signal between points that in S are at a distance a apart may be described as starting at (t, x, y, z) , going to

$$\left(t + \frac{a}{c-v}, x + \frac{ca}{c-v}, y, z \right),$$

then to

$$\left(t + \frac{2ca}{c^2 - v^2}, x + \frac{2cva}{c^2 - v^2}, y, z \right);$$

or more conveniently, by the use of the relative abscissa $x' = x - vt$, as involving successively

$$[t, x', y, z], \quad \left[t + \frac{a}{c-v}, x' + a, y, z \right],$$

and

$$\left[t + \frac{2ca}{c^2 - v^2}, x', y, z \right];$$

a backward longitudinal signal over an object of the same apparent length is like this except that (a, c) are replaced by $(-a, -c)$. Since the first and third are spatially coincident in Σ the intermediate τ must be the arithmetic mean of the others. Hence, if $\tau = F(x', t)$, where for brevity (y, z, v) are omitted temporarily from the functional symbol because only parameters, the conditions are the Einstein functional equation and its mate for opposite signals:

$$\begin{aligned}
 (1) \quad & F(x', t) + F\left(x', t + \frac{2ca}{c^2 - v^2}\right) \\
 & = 2F\left(x' + a, t + \frac{a}{c - v}\right) = 2F\left(x' - a, t + \frac{a}{c + v}\right).
 \end{aligned}$$

The combined or right hand equality says in effect that $F(\alpha, \beta)$ is invariant under the transformation

$$\alpha' = \alpha + 2a, \quad \beta' = \beta + \frac{2va}{c^2 - v^2},$$

with a arbitrary, and is therefore a function of the invariant

$$\beta - \frac{v\alpha}{c^2 - v^2}$$

only, so that

$$F(x', t) = F\left(t - \frac{vx'}{c^2 - v^2}\right).$$

Substitution in the first equality in (1) reduces it to the form

$$F(u) + F(u + 2h) = 2F(u + h),$$

so that F , being continuous, is linear. Hence

$$(2) \quad \tau = \frac{\varphi(y, z, v)}{\kappa} \left(t - \frac{vx}{c^2}\right) + f(y, z, v),$$

where

$$\kappa^2 = 1 - \frac{v^2}{c^2}.$$

Here the first coefficient is written in a form later convenient, the φ corresponding to Einstein's.

A lateral signal over an object of width b in a plane of constant z would involve

$$(t, x, y, z), \quad \left(t + \frac{b}{c\kappa}, x + \frac{vb}{c\kappa}, y + b, z\right),$$

and

$$\left(t + \frac{2b}{c\kappa}, x + \frac{2vb}{c\kappa}, y, z\right).$$

The condition that here also the intermediate τ be the arithmetic mean reduces to

$$\left[t - \frac{vx}{c^2} + \frac{b}{c} \right] [\varphi(y, z, v) - \varphi(y + b, z, v)] \\ + [f(y, z, v) - f(y + b, z, v)] = 0,$$

showing that φ and f must be independent of y and by symmetry of z also.

If the lateral signal be considered first, and $\tau = F(x', y, t)$, then the corresponding condition is

$$(3) \quad F(x', y, t) + F\left(x', y, t + \frac{2b}{c\kappa}\right) \\ = 2F\left(x', y + b, t + \frac{b}{c\kappa}\right) = 2F\left(x', y - b, t + \frac{b}{c\kappa}\right),$$

which shows that τ must be independent of y , and similarly of z , and must be linear in t , so that

$$(4) \quad \tau = \psi(x', v)t + g(x', v).$$

Then the longitudinal signal gives the condition

$$t[\psi(x', v) - \psi(x' + a, v)] \\ + \frac{a}{c - v} \left[\frac{c}{c + v} \psi(x', v) - \psi(x' + a, v) \right] \\ + [g(x', v) - g(x' + a, v)] = 0$$

which shows that ψ must be independent of x' and that

$$(5) \quad g(x' + a, v) - g(x', v) = -\frac{av}{c^2 - v^2} \psi(v),$$

so that $g(x', v) + vx'\psi(v)/(c^2 - v^2)$ must be a function of v only. The two orders of proof thus lead to the result

$$(6) \quad \tau = \frac{\varphi(v)}{\kappa} \left(t - \frac{vx}{c^2} \right) + f(v),$$

which evidently contains all the information that the signalling process can give about the time reckoning in Σ , since the undetermined functions f and φ correspond to arbitrary choice of origin and units. This equation shows that the time intervals in Σ of one-way transit are $a\varphi(v)/\kappa c$ and $b\varphi(v)/c$ for the longitudinal and lateral signals respectively.

The condition that in each of the instances mentioned of

to-and-fro signal the initial and final space points coincide in Σ shows that (ξ, η, ζ) are functions of (x', y, z, v) only, independent of t ; which is in fact equivalent to the condition of rigid motion of Σ with respect to S , when the time reckoning in Σ is given by (6). The relation between the two systems of space coordinates will appear from consideration of a one-way signal of arbitrary direction and time interval. For convenience let $x'/\kappa = x''$; then in general

$$\Delta x = \kappa \Delta x'' + v \Delta t, \quad \Delta \tau = \varphi(v)(\kappa \Delta t - v \Delta x''/c^2).$$

From these it follows that the distance r , which must be the same as $c\Delta t$, travelled in the time interval Δt by the signal in (x, y, z) space, is related to the apparent distance r'' in the space of the auxiliary coordinates (x'', y, z) by

$$r^2 = c^2(\Delta t)^2 = r''^2 - v^2(\Delta x'')^2/c^2 + v^2(\Delta t)^2 + 2\kappa v \Delta x'' \Delta t,$$

which reduces to

$$r''^2 = (\kappa c \Delta t - v \Delta x'')^2 = c^2(\Delta \tau)^2/\varphi^2.$$

This shows that the distance in (ξ, η, ζ) space is always the same as in $(\varphi x'', \varphi y, \varphi z)$ space, so that these two triples must be linear orthogonal transforms of each other; the known proof of this requires no assumption of differentiability. Combined with (6) this gives essentially Einstein's result, except that there is here no initial supposition as to the orientation of the axes in Σ .

If, however, the special supposition be made that ξ is a function of (x, t) only, therefore of (x', t) only, then the longitudinal signal gives

$$(7) \quad G\left(x', t + \frac{2ca}{c^2 - v^2}\right) = G(x', t),$$

making G independent of t ; also

$$(8) \quad G(x' + a) = G(x') + a\varphi(v)/\kappa,$$

so that, G being continuous, $\xi = \varphi(v)x'/\kappa + g(v)$. Then the remaining conditions show that the pair (η, ζ) is an orthogonal transform of $(\varphi y, \varphi z)$. The general result is conveniently viewed from Minkowski's point of view, as giving $(ic\tau, \xi, \eta, \zeta)$ from (ict, x, y, z) by a rigid rotation about an arbitrary origin, followed by a magnification with factor φ , whose arbitrariness indicates the freedom remaining in the choice of units.

Thus far the proof refers to a particular pair of systems only, and the undetermined functions φ , f , g , and the like, instead of being written as functions of v , are thus more suitably to be regarded as merely functions of the pair of systems whose relation is sought. For there is nothing in the general assumptions to imply that their values need be the same for two Σ 's moving with respect to S with the same velocity, even if the directions were also the same, in which case they would be relatively to each other at rest. But in the case of a set of systems, each standing to each in the relation restricted by the general postulates, certain conditions are imposed on these functions by the requirement of transitivity, or the group property as it here becomes. For example, with n systems there would be $n(n-1)/2$ factors φ and their inverses, but only $(n-1)$ arbitrary ratios of units. If then instead of $\varphi(v)$ there be written $\varphi(\Sigma, S)$, which for this purpose might be more suggestively $\varphi(\Sigma/S)$, these conditions have the form

$$(9) \quad \varphi(U, T)\varphi(T, S) = \varphi(U, S).$$

The algebraic form of the implications of this equation would depend on the set of systems considered. If the artificiality be discarded that would allow φ to be different for two Σ 's at rest with respect to each other or moving with respect to S with the same speed in different directions, the condition reduces to

$$(10) \quad \varphi(v)\varphi(v') = \varphi(v''),$$

where v'' is the resultant of v and v' in Einstein's sense. If any rigid translation with respect to any S gives a possible Σ , then, by alteration of the direction of v' , v'' can vary while v and v' are constant, so that φ must be constant and therefore unity, as Einstein makes it. Poincaré* had already indicated this as a condition for a group in the Lorentz transformation.† If, however, only parallel motions were admitted, and v taken as with sign, the condition would be

$$(11) \quad \varphi(v)\varphi(v') = \varphi\left(\frac{v+v'}{1+vv'/c^2}\right).$$

The continuous solution of this, readily obtained through the transformation $v/c = \tanh \alpha$, is

* Poincaré, *Comptes Rendus*, vol. 140, pp. 1504-1508 (1905).

† Lorentz, *Proc. Amsterdam A.*, vol. 6, pp. 809-831 (1904); and *Theory of Electrons*, p. 197.

$$(12) \quad \varphi(v) = \left(\frac{c+v}{c-v} \right)^n,$$

where n is an arbitrary constant. Einstein's condition that reversal of v leave φ unaltered makes $n = 0$; as to physical interpretation it may be noted that this is the only case where the contraction factor would have an expansion in powers of v free from a linear term.

Einstein's original proof referred only to signals transmitted through a vacuum. The case of a refracting medium of arbitrary index μ , at rest with respect to Σ , was considered by Laub,* who imposed the requirement that the comparison of space and time reckoning be independent of the value of μ . He obtained thereby the various velocities of propagation with respect to S , leading to the Fresnel-Fizeau convection coefficients. These results were then shown by Laue† to be instances of the Einstein law of composition of velocities. The following examples illustrate the modification of the functional equations to suit this case.

The medium is supposed to appear isotropic in Σ , with velocity of propagation c/μ ; but in S light will have different velocities in different directions, except when μ is unity. Considering only the longitudinal signal let the forward and backward velocities in S be c' , c'' . Then (1) is replaced by

$$(13) \quad \begin{aligned} F(x', t) + F\left(x', t + a \left[\frac{1}{c' - v} + \frac{1}{c'' + v} \right] \right) \\ = F\left(x' + a, t + \frac{a}{c' - v}\right) \\ = F\left(x' - a, t + \frac{a}{c'' + v}\right), \end{aligned}$$

making $F(\alpha, \beta)$ invariant under the transformation

$$\alpha' = \alpha + 2a, \quad \beta' = \beta + a \left[\frac{1}{c' - v} - \frac{1}{c'' + v} \right],$$

so that

$$F(x', t) = F\left(t - \frac{x'}{2} \left[\frac{1}{c' - v} - \frac{1}{c'' + v} \right] \right).$$

Linearity appearing as before, the result is

* Laub, *Ann. d. Phys.*, vol. 23, pp. 738-744 (1907).

† Laue, *Ann. d. Phys.*, vol. 23, pp. 989-990 (1907).

$$(14) \quad \tau = \frac{\varphi(v, \mu)}{\kappa} \left(t - \frac{x'}{2} \left[\frac{1}{c' - v} - \frac{1}{c'' + v} \right] \right) + f(v, \mu).$$

Equation (7) is replaced by

$$(15) \quad G(x', t) = G \left(x', t + a \left[\frac{1}{c' - v} + \frac{1}{c'' + v} \right] \right),$$

the equation (8) by

$$(16) \quad G(x' + a) = G(x') + \frac{a\varphi a}{2\mu\kappa} \left[\frac{1}{c' - v} + \frac{1}{c'' + v} \right],$$

whence

$$(17) \quad \xi = \frac{c\varphi x'}{2\mu} \left[\frac{1}{c' - v} + \frac{1}{c'' + v} \right] + g(x, \mu).$$

The corresponding modifications for lateral and other signals are readily made.

It appears therefore that Einstein's assumption of differentiability is unnecessary, and that the needed features of "homogeneity" are already implicit in the optical and kinematic postulates as here interpreted.

THE UNIVERSITY OF CHICAGO,
April 8, 1919.

FORMULAS FOR CONSTRUCTING ABRIDGED MORTALITY TABLES FOR DECENNIAL AGES.

BY PROFESSOR C. H. FORSYTH.

MR. GEORGE KING presented in the British Registrar-General's Report for 1914 a method of constructing abridged mortality tables which consist merely of the values appearing in an ordinary mortality table but corresponding only to each quinquennial age. The computation of the values corresponding to the other ages is eliminated by the method and an enormous saving of labor is thus gained.

Such tables promise to prove very useful in investigations in human vitality because many problems which have heretofore involved too much computation to permit much investigation now become relatively easy. For this reason it seems

well worth while to derive and present the analogous formulas for constructing mortality tables for *decennial* ages.

Tables based on decennial ages require little more than half the computation necessary for tables based on quinquennial ages after the proper statistical data are collected and properly arranged,—the computation takes little over an hour even when the incidental computation is performed by hand—give practically the same information and involve much more suitable ages (i.e., ages 10, 20, 30, etc., instead of 12, 17, 22, etc.).

First, the given population statistics and the mortality statistics are each easily arranged in the age groups 5-14,* 15-24, 25-34, etc. Then, the following formula† to second differences‡ used for interpolating ordinates among areas

$$(1) \quad u_{x/t} = w_0 + (2x - t + 1) \frac{\Delta w_0}{2t^2} \\ + \{3x^2 + 3x(1 - 2t) + (1 - 3t + 2t^2)\} \frac{\Delta^2 w_0}{6t^3},$$

where

$$w_{x/t} = u_{x/t} + u_{(x+1)/t} + \cdots + u_{(x+t-1)/t}$$

and $w_{x/t}$ is generally abbreviated to w_{nx} , is applied to determine the population and the deaths for a suitable individual age. Thus, if w_0 , w_1 , and w_2 refer to the populations (or deaths) for the age groups 5-14, 15-24, and 25-34 respectively, $u_{x/t}$ for $t = 10$ and $x = 15$ gives the population (or deaths) for the individual age 20, and

$$(2) \quad u_{15/10} = .1(w_0 + \Delta w_0) + \frac{.01}{2} (\Delta w_0 - .3\Delta^2 w_0) \\ = .1w_1 + \frac{.01}{2} (\Delta w_0 - .3\Delta^2 w_0)$$

and similarly for ages 30, 40, etc.

Knowing the population L_x and the number of deaths $\bar{d}_x$ at an individual age x , the value of the probability of living one year or p_x is obtained by the well-known formula

* The great variation and uncertainties of the death rate at ages in the neighborhood of birth necessitate the omission of the age group 0-5.

† *Quarterly Publications of the American Statistical Association*, Dec. 1916.

‡ It is generally accepted that second differences are sufficient when dealing with ordinary statistical data.

$$(3) \quad p_x = 1 - q_x = \frac{L_x - \frac{1}{2}d_x}{L_x + \frac{1}{2}d_x}.$$

Probably, a better plan for computing values of p_x is to combine formulas (2) and (3) to give

$$p_{x+5} = \frac{.1(W_x - w_x) + \frac{.01}{2} \{(\Delta W_{x-10} - \Delta w_{x-10}) - .3(\Delta^2 W_{x-10} - \Delta^2 w_{x-10})\}}{.1(W_x + w_x) + \frac{.01}{2} \{(\Delta W_{x-10} + \Delta w_{x-10}) - .3(\Delta^2 W_{x-10} + \Delta^2 w_{x-10})\}},$$

where W_x and w_x refer to the population and *one-half* the deaths respectively, corresponding to the decennial age group running from x to $x + 9$ and the subscripts are changed to a more workable form to correspond to the ages. If we write D_x for $W_x - w_x$ and S_x for $W_x + w_x$ and notice that

$$\Delta^* W_x \pm \Delta^* w_x = \Delta^*(W_x \pm w_x) = \Delta^* S_x \quad \text{or} \quad \Delta^* D_x$$

the formula given above reduces to

$$(4) \quad p_{x+5} = \frac{D_x + \frac{.1}{2}(\Delta D_{x-10} - .3\Delta^2 D_{x-10})}{S_x + \frac{.1}{2}(\Delta S_{x-10} - .3\Delta^2 S_{x-10})}.$$

If we let $t = 10$ and $x = 5$ in formula (1), we obtain in like manner

$$(4') \quad p'_{x+5} = \frac{D_x + \frac{.1}{2}\Delta D_x - .06\frac{1}{2}\Delta^2 D_x}{S_x + \frac{.1}{2}\Delta S_x - .06\frac{1}{2}\Delta^2 S_x},$$

which may be applied to the age groups 5-14, 15-24, and 25-34 to give p_{10} .

To apply formulas (4) and (4'), columns of values of D_x and S_x are first obtained by inspection from columns of values of W_x and w_x ; the divisions are then performed by logarithms to give a column of values of $\log p_x$.

Values of $\log_{10} p_x$ are next obtained from values of $\log p_x$ as follows: If we sum the ten expressions obtained by sub-

stituting $10/10$, $11/10$, $\dots$, $19/10$ in Newton's formula (to second differences)

$$u_x = u_0 + x\Delta u_0 + \frac{x(x-1)}{2} \Delta^2 u_0,$$

we obtain

$$w_1 = 10u_0 + 14\frac{1}{2}\Delta u_0 + 3\frac{2}{3}\Delta^2 u_0,$$

which may be written

$$(5i) \quad w_1 = 10u_1 + 4\frac{1}{2}\Delta u_0 + 3\frac{2}{3}\Delta^2 u_0,$$

where, however, the true coefficient of $\Delta^2 u_0$ is 3.675. The arbitrary change in coefficients simplifies the computation considerably while the error thus introduced will generally prove insignificant in the final results.

Since $\log_{10} p_x = \log p_x + \log p_{x+1} + \dots + \log p_{x+9}$, formula (5i) enables one to obtain a column of values $\log_{10} p_{20}$, $\log_{10} p_{30}$, etc., from a column of values $\log p_{10}$, $\log p_{20}$, etc. The value of $\log_{10} p_{10}$ is obtained by using the formula

$$(5i') \quad w_0 = 10u_0 + 4\frac{1}{2}\Delta u_0 + .8\frac{1}{4}\Delta^2 u_0,$$

which is the sum of the ten expressions obtained by substituting $0/10$, $1/10$, $\dots$, $9/10$ in Newton's formula.

Assuming a suitable radix or number of individuals living at age 10, the values of $\log_{10} p_x$ are added accumulatively to the logarithm of the radix to give in succession the logarithm of the number of survivors at each subsequent tenth age.

The radix is usually assumed to be 100,000 at age 10 but after considerable experience with the difficulties encountered in trying to complete such mortality tables at the higher ages the writer has become convinced that the radix of at least an *abridged* mortality table should not ordinarily be greater than 1,000 at age 10. Even then in a few exceptional cases it has been found necessary to adopt rather arbitrary measures to complete the tables at the higher ages. As the values corresponding to the highest ages have very little effect on the values at the earlier ages, the adoption of a radix of 1,000 obviates the necessity of introducing arbitrary and often questionable measures for *unimportant details*.

The column of survivors for decennial ages obtained in the manner just explained constitutes the abridged mortality table desired, but a column of expectations of life is always desirable and is obtained as follows. If we sum the expressions obtained by substituting $11/10$, $12/10$, $\dots$, $20/10$ in Newton's formula, we get

$$(5t) \quad w_{11/10} = 10u_1 + 6\frac{1}{2}\Delta u_0 + 4\frac{2}{3}\Delta^2 u_0,$$

where the coefficient $4\frac{2}{3}$ is written instead of the true value 4.675 to simplify computation.

If formula (5t) is applied to the column of survivors 1_{10} , 1_{20} , etc., constituting the abridged mortality table, we obtain the number of survivors corresponding to the age groups 20–29, 30–39, etc. The number of survivors corresponding to the age group 10–19 is obtained by applying the formula

$$(5t') \quad w_{1/10} = 10u_0 + 5\frac{1}{2}\Delta u_0 + .8\frac{1}{4}\Delta^2 u_0,$$

which is the sum of the expressions obtained by substituting $1/10$, $2/10$, ..., $10/10$ in Newton's formula.

By way of distinction, formulas (5i) and (5i') have been called *initial* forms and formulas (5t) and (5t') *terminal* forms, for obvious reasons.

If, now, the column of survivors for the decennial age groups be summed accumulatively, beginning at the highest ages and the successive sums be divided by the number of survivors at the individual age just preceding in each case—as given in the abridged mortality table—a column of values of *curtate* expectation of life is obtained. Assuming, however, that the average individual lives a half year in the year of his death, it is customary to add 0.5 to each of the values just mentioned to obtain values of the *complete* expectation of life—the values usually quoted as the expectation of life.

In computing the expectation of life there is no satisfactory way of determining the number of survivors corresponding to the last age group. This fact should give no serious concern, for beyond much doubt the errors in the data at the higher ages—especially those due to the consistent overstatement of ages—render worthless any special efforts to be accurate at these ages. If only one decimal place is kept in the values of the expectation of life, it will make no difference at all what the number of survivors corresponding to the last age group is taken to be, because it will not affect any of the values of the expectation of life except those at the very highest—say, two or three—decennial ages and these values because of their inaccuracy should not be given very much weight.

The values of the death rates q_x are obtained from the values of p_x by means of the relation between q_x and p_x given in formula (3).

DARTMOUTH COLLEGE.

SHORTER NOTICES.

Projective Geometry. By L. WAYLAND DOWLING. McGraw Hill Book Company, 1917. xiii + 215 pp.

THIS book (as the preface tells us) has been developed from a course of lectures given by the author for a number of years at the University of Wisconsin.

In its treatment of the subject it follows closely the main lines of the classical text of Reye in nomenclature and in point of view. It begins with an introductory chapter giving definitions of the elements, projection, section, and ideal elements. The point range, sheaf of lines, sheaf of planes, field of points, field of lines, bundle of lines, bundle of planes, space of points, space of planes, special linear complex, and space of lines are named as the eleven primitive forms. No formal definition of primitive form is given which would enable the reader to judge why exactly these eleven forms are given and others are excluded.

The concept of motion of a line about a point and that of parallelism are made use of in defining an ideal point, and this is followed by the "fundamental assumption: On every straight line there is one and only one ideal or infinitely distant point. This point makes the line continuous from any one point on it to any other point on it in either direction. Through a given point there can be drawn one and only one line parallel to a given line. This parallel intersects the given line in the ideal or infinitely distant point."

This quotation perhaps shows the point of view of the author in basing his projective geometry on a completely developed euclidean geometry. Duality is introduced by means of illustrations followed by a statement of the method of obtaining new theorems from others by interchanging elements and their reciprocals in the statements. Neither the logical bearing of the principle nor the extent of its validity is discussed. The study of the complete quadrangle leads up to harmonic forms and the cross-ratio of a harmonic range is shown to be -1 by the use of ordinary metrics. From harmonic ranges harmonic scales are developed. The harmonic separation theorems are obtained from a free and intuitive use of continuity. A form of the Dedekind postulate is

introduced using the concepts of segments and order as modified by the introduction of ideal elements. The proof of the fundamental theorem of von Staudt is then based on continuity. Curves and envelopes of the second order, poles and polars with respect to a curve are then studied in considerable detail, with the use of metrics for some properties.

An involution on a form is defined as a cyclic projectivity of order 2 and conjugate imaginary points, lines and planes are introduced from the elliptic involution. An entire chapter is devoted to the focal properties of conics. Collineations, dualities, affinities, and polarities are treated briefly.

The figures are fairly well done, the typography is very good, and the volume as a whole is neat and attractive. The large number of exercises scattered throughout the book add much to its utility as a textbook. The little historical notes although very brief are also valuable and stimulating.

In the opinion of the reviewer, such a book as this one, which is avowedly not concerned with the more logical phases of the subject, would be more valuable to many if it gave at least references to such books as that of Veblen and Young, where such treatment could be found. On the whole, however, the book will no doubt be of much service in beginning courses in the subject.

F. W. OWENS.

Introduction to the Elementary Functions. By RAYMOND BENEDICT MCCLENON, with the editorial cooperation of WILLIAM JAMES RUSK. Boston, Ginn and Company, 1918. x + 244 pp.

UNDER the above title the authors present in book form their idea of the content of a required course in mathematics for freshmen in a small college. As presented, the text is the result of five years of teaching of the course in such a college. The elementary functions are those from plane trigonometry, plane analytic geometry, and the elements of differential calculus. These subjects are combined with such review topics from algebra as is necessary to bind the topics presented into an organic whole. The chapters are not sections taken from a single one of the principal subjects, but are mixtures. The calculus covers only the differentiation of algebraic functions with applications to rate and maxima and minima problems.

The treatment of topics is informal and inductive, there being a noticeable absence of formal proofs. Naturally the reviewer does not agree with the arrangement of material nor with its content, but we do agree with an avowed object of the book, namely that of coaxing students in a small college, or elsewhere for that matter, into a further study of non-required mathematics. We object to the form of the discussion of the quadratic in one unknown (pages 46, 47) wherein the authors seem to discard complex roots, as *roots*. Of course problems in analytic geometry and elsewhere lead to equations with complex roots, which are not interpretable in terms of the real elements presented in the problem. But this does not seem to be the meaning of the authors.

C. F. CRAIG.

NOTES.

THE following mathematical papers have appeared in recent numbers of the *Proceedings of the National Academy of Sciences*: volume 5, number 1 (January, 1919): "A theorem in power series, with an application to conformal mapping," by T. H. GRONWALL; number 3 (March): "Tables of the zonal spherical harmonic of the second kind $Q_1(z)$ and $Q_1'(z)$," by A. G. WEBSTER and W. FISHER; number 4 (April): "On the real folds of abelian varieties," by S. LEFSCHETZ; "Covariants of binary modular groups," by O. E. GLENN; "The general solution of the indeterminate equation: $Ax + By + Cz + \dots = r$," by D. N. LEHMER; number 6 (June): "On the most general class L of Fréchet in which the Heine-Borel-Lebesgue theorem holds true," by R. L. MOORE; "On a certain class of rational ruled surfaces," by A. EMCH; number 7 (July): "On the twist in conformal mapping," by T. H. GRONWALL; "Groups involving only two operators which are squares," by G. A. MILLER; "Real hypersurfaces contained in abelian varieties," by S. LEFSCHETZ.

THE concluding (June) number of volume 20 of the *Annals of Mathematics* contains the following papers: "Relations between abstract group properties and substitution groups," by G. A. MILLER; "The complete quadrilateral," by J. W.

CLAWSON; "Triply conjugate systems with equal point invariants," by L. P. EISENHART; "On a system of linear partial differential equations of the hyperbolic type," by T. H. GRONWALL; "Some properties of circles and related conics," by J. H. WEAVER; "Integrals in an infinite number of dimensions," by P. J. DANIELL; "On the differentiability of the solution of a differential equation with respect to a parameter," by J. F. RITT; "Note on the derivatives with respect to a parameter of the solutions of a system of differential equations," by T. H. GRONWALL; "On quaternions and their generalization and the history of the eight-square theorem. Addenda," by L. E. DICKSON.

The July number (volume 41, number 3) of the *American Journal of Mathematics* contains the following papers: "Invariants of differential geometry by the use of vector forms," by C. D. RICE; "On certain saltus equations," by HENRY BLUMBERG; "Investigations on the plane quartic," by TERESA COHEN; "On surfaces containing two pencils of cubic curves," by C. H. SISAM; "Modular invariants of a quadratic form for a prime power modulus," by J. E. MCATEE.

ON November 2, 1918, the Norwegian mathematical society was founded at Christiania. Its officers are Professors STÖRMER, president, BIRKELAND, vice-president, PALMSTRÖM, secretary, and SOLBERG, treasurer. The Society will publish a *Bulletin*, under the editorship of Professors Alexander and Heegaard.

THE firm of Julius Springer Berlin, announces the publication of a new journal devoted exclusively to original mathematical memoirs, the *Mathematische Zeitschrift*. It is edited by Professor L. Lichtenstein, with the collaboration of Professors K. Knopp, E. Schmidt, and I. Schur and an editorial committee consisting of Professors W. Blaschke, L. Féjer, C. Herglotz, A. Kneser, E. Landau, O. Perron, F. Schur, E. Study and H. Weyl. Two volumes appear annually.

ON December 12, 1918, Professor FELIX KLEIN celebrated the fiftieth anniversary of his doctorate, and on April 25, 1919, his seventieth birthday.

PROFESSOR C. KOSTKA, of Instertburg, Dr. W. LIETZMANN, of

Jena, Professor H. E. TIMERDING, of the Braunschweig technical school, and Professor W. LOREY, of Leipzig, have been elected members of the Leopoldinisch-Carolinische Akademie der Naturforscher, of Halle.

PROFESSOR E. GRÜNEISEN has been appointed associate professor of mathematical physics at the University of Marburg.

PROFESSOR G. HESSENBERG, of the Breslau technical school, has been made Geheimer Regierungsrat.

PROFESSOR ERHARD SCHMIDT, of the University of Berlin, has been elected a member of the mathematico-physical class of the academy of sciences of Berlin.

DR. A. TIMPE, of the University of Münster, has been appointed professor of mathematics at the agricultural school of Berlin.

PROFESSOR N. E. NÖRLUND, of the University of Lund, has been appointed professor of mathematics at the University of Copenhagen.

PROFESSORS W. FEUSSNER, of the University of Marburg, and G. FREGE, of the University of Jena, have retired from active teaching.

DR. E. JACOBSTHAL, of the Berlin technical school, has been promoted to a professorship of mathematics.

PROFESSOR H. VON MANGOLDT has been elected president of the Deutsche Mathematiker-Vereinigung for 1918-19. Professor FELIX KLEIN has been elected honorary president for the same period, in honor of the fiftieth anniversary of his doctorate.

PROFESSOR H. FEHR, of the University of Geneva, has been elected foreign correspondent of the Royal society of sciences of Liège.

DR. A. MOHRMANN has been appointed professor of mathematics at the University of Basle.

PROFESSOR P. APPELL has retired from the office of dean of the faculty of sciences of the University of Paris, after sixteen years of service.

PROFESSOR H. ANDOYER, of the Sorbonne, has been elected member of the Paris academy of sciences, in the section of astronomy.

DR. H. LEBESGUE, of the University of Paris, has been promoted to a professorship of the application of analysis to geometry. Professor Lebesgue has been elected president of the Société mathématique de France for the year 1919.

DR. E. VESSIOT has been appointed professor of general mathematics at the University of Paris, as successor to Professor C. GUICHARD, who has been transferred to a professorship of higher geometry.

DR. J. CHAZY has been appointed professor of mathematics at the University of Lille, as successor to the late Professor J. CLAIRIN.

DR. P. HUMBERT has been appointed maître de conférences at the University of Montpellier, during the absence of Dr. A. DENJOY, who is at the University of Utrecht.

DR. L. ROY has been appointed professor of analytical mechanics at the University of Toulouse, as successor to the late Professor S. LATTÈS.

DR. H. R. HASSÉ, of the University of Manchester, has been appointed professor of mathematics at the University of Bristol.

At the May meeting of the American academy of arts and sciences, Professors JOSEPH LIPKA, G. A. MILLER, F. R. MOULTON, and VIRGIL SNYDER were elected fellows in the section of mathematics and astronomy.

MAJOR E. V. HUNTINGTON, who has been assigned to statistical duty in Washington, D. C., during the past year, was recently made a member of the General Staff. He is returning to Harvard University in the autumn.

DR. HENRY BLUMBERG, of the University of Illinois, has been promoted to an assistant professorship of mathematics.

ASSISTANT professor W. W. RANKIN, of the University of North Carolina, has been granted leave of absence for the year 1919-20 and has been appointed instructor in mathematics at Columbia University for that period.

PROFESSOR M. E. GRABER, of Heidelberg University, Tiffin, Ohio, has been appointed professor of physics at Morningside College, Sioux City, Iowa.

ASSISTANT professor M. O. TRIPP, of the University of Maine, has been promoted to an associate professorship of mathematics.

PROFESSOR W. GROSS, of the University of Vienna, died October 29, 1918, at the age of thirty-two years.

THE death is reported of Dr. J. KELLER, of the technical school at Zürich.

PROFESSOR C. BRANDENBERGER, of the cantonal school at Zürich, died January 2, 1919.

PROFESSOR PAUL MANSION, of the University of Ghent, died April 18, 1919, at the age of seventy-five years.

PROFESSOR C. ALASIA DE QUESADA, of the gymnasium at Albengo, died November 19, 1918, at the age of forty-nine years.

PROFESSOR EMILE DUMONT, of the Institut Michot-Montgenast at Brussels, was killed in action in the late war.

THE Rt. Hon. JOHN WILLIAM STRUTT, Lord Rayleigh, died July 1, 1919, at the age of seventy-six years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- BORTOLOTTI (E.). Italiani scopritori e promotori di teorie algebriche. Modena, O. Ferraguti, 1919. 8vo. 102 pp.
- BOUMAN (P. J.) en ZELM (J. C. van). De rekenkundige denkbaarheden in logischen samenhang, met, als proeve van toegepaste logica, een rekenmethode voor de lagere school. Amsterdam, Versluys, 1918. 8vo. 381 pp. Fl. 2.90
- BOUTROUX (P.). Les principes de l'analyse mathématique. Exposé historique et critique. Tome 2. Paris, Hermann, 1919. 8vo. 512 pp. Fr. 20.00
- BRENDEL (M.). See KLEIN (F.).
- CAJORI (F.). A history of mathematics. Second edition revised and enlarged. New York, Macmillan, 1919. 8vo. 10 + 514 pp. \$4.00
- KLEIN (F.), BRENDL (M.) und SCHLESINGER (L.). Materialien für eine wissenschaftliche Biographie von Gauss. Heft 6: Die Wechselwirkung zwischen Zahlenrechnen und Zahlentheorie bei C. F. Gauss, von Ph. Maennchen. Leipzig, Teubner, 1918. 8vo. 47 pp. M. 3.00
- LENGAUER (J.). Analytische Geometrie. 2te Auflage. Kempten, Kösel, 1918. 113 pp.
- MAENNCHEN (P.). See KLEIN (F.).
- MANNOURY (G.). Over de sociale betekenis van de wiskundige denkvorm. Groningen, Noordhoff, 1917. 8vo. 23 pp. Fl. 0.60
- OPFERMANN (A.). Premiers éléments d'une théorie du quadrilatère complet. Paris, Gauthier-Villars, 1919. 8vo. 76 pp. Fr. 4.00
- SCHLESINGER (L.). See KLEIN (F.).
- SCHUH (F.). Leerboek der theoretische rekenkunde. Eerste deel. Groningen, Noordhoff, 1919. 8vo. 452 pp. Fl. 8.50
- TEIXEIRA DE MATTOS (J.). Over satellietruimten en eenige toepassingen op satellietopperplakken. (Diss.) Amsterdam, H. G. van Dorsten. 8vo. 8 + 125 pp.
- VOLLENHOVE (D. H. T.). De wijsbegeerte der wiskunde van theistisch standpunt. (Diss.) Amsterdam, Wed. G. van Sost, 1918. 8vo. 15 + 444 pp.
- ZELM (J. C. van). See BOUMAN (P. J.).

II. ELEMENTARY MATHEMATICS.

- INGOLD (L.). See KENYON (A. M.).
- KARPINSKI (L. C.). Four-place logarithmic and trigonometric tables, together with interest tables. Ann Arbor, George Wahr, 1918. 30 pp. Paper. \$0.30

- KENYON (A. M.) and INGOLD (L.). Elements of plane trigonometry, with brief tables. New York, Macmillan, 1919. 12mo. 9 + 117 + 18 + 12 pp. \$1.00
- . Elements of plane trigonometry, with complete tables. New York, Macmillan, 1919. 12mo. 9 + 117 + 18 + 124 pp. \$1.20
- LENNES (N. J.). See SLAUGHT (H. E.).
- LIETZMANN (W.). See SCHUSTER (M.).
- LIETZMANN (W.) und ZÜHLKE (P.). Aufgabensammlung und Leitfaden für Arithmetik, Algebra und Analysis. Ausgabe B für Realanstalten, Oberstufe. Leipzig, Teubner, 1918. 334 pp.
- . Geometrische Aufgabensammlung. Ausgabe A für Gymnasien, Oberstufe. Leipzig, Teubner, 1918. 150 pp.
- LIGHTFOOT (J.). An elementary algebra for junior students. London, Russell, 1919. 3s. 6d.
- . Graphic algebra for elementary and intermediate students. London, Russell, 1919. 1s.
- MYERS (G. W.). See PALMER (C. I.).
- PALMER (C. I.) and TAYLOR (D. P.). Solid geometry. Edited by G. W. Myers. Chicago, Scott, Foresman and Company, 1918. 6 + 178 pp.
- ROBERTSON (J.). Dictionary for international commercial quotations to translate units of value from one currency, weight and measure direct into another. London, Oxford University Press, 1918. 8vo. 12 + 208 pp.
- SCHUSTER (M.). Geometrische Aufgaben, herausgegeben von W. Lietzmann. Leipzig, Teubner, 1918. Ausgabe A (Vollanstalten), 3ter Teil: Stereometrie. 3te Auflage. Ausgabe B (Progymnasien und Realschulen): Planimetrie. 4te Auflage. 103 + 118 pp. Geb. M. 2.60 + 2.20
- SLAUGHT (H. E.) and LENNES (N. J.). Solid geometry with problems and applications. Revised edition. Boston, Allyn and Bacon, 1919. 6 + 211 pp. \$1.00
- TAYLOR (D. P.). See PALMER (C. I.).
- WATSON (B. M.) and WHITE (C. E.). Modern arithmetic. New York, Heath, 1918. Primary, 8 + 252 pp. Intermediate, 10 + 254 pp. For upper grades, 9 + 302 pp.
- WHITE (C. E.). See WATSON (B. M.).
- YOUNG (J. W. A.). The teaching of mathematics in the elementary and secondary school. New Impression. New York, Longmans, 1919. 8vo. 8 + 491 pp. \$1.75
- ZÜHLKE (P.). See LIETZMANN (W.).

III. APPLIED MATHEMATICS.

- BLANCHARD (A. H.). American highway engineers' handbook. By A. H. Blanchard and seventeen associate editors. New York, Wiley, 1919. 25 + 1658 pp. \$5.00
- BRESTER (A.). A summary of my theory of the sun. The Hague, W. P. Stockum, 1919. 62 pp.

BUE (A. DEL). *Lezioni di meccanica generale. Parte 2: Dinamica.* Città di Castello, Soc. tip. Leonardo da Vinci, 1919. 8vo. 171 pp.
L. 5.50

COMSTOCK (G. C.). *The Sumner line or line of position as an aid to navigation.* New York, Wiley, 1919. 6 + 70 pp. 12mo. \$1.25

FRENCH (J. W.). See STEINHEIL (A.).

FULLER (C. E.) and JOHNSTON (W. A.). *Applied mechanics. Volume 2: Strength of materials.* New York, Wiley, 1919. 11 + 556 pp. \$3.75

GARUFFA (E.). *L'aviazione: aeroplani, idrovolanti, eliche. 2a edizione, interamente rifatta e aumentata. (Manuali Hoepli.)* Milano, Hoepli, 1919. 24mo. 23 + 915 pp. L. 20.00

———. *Turbine a vapore, con un capitolo sulle turbine a gas: teoria, calcolazione, costruzioni. (Manuali Hoepli.)* Milano, Hoepli, 1919. 24mo. 15 + 782 pp. + tavola. L. 22.50

HARTMANN (O.). *Astronomische Erdkunde. 5te Auflage.* Stuttgart, Grub, 1918. 83 pp.

JOHNSTON (W. A.). See FULLER (C. E.).

LORIA (G.). *Metodi de geometria descrittiva. 2a edizione, riveduta e migliorata. (Manuali Hoepli.)* Milano, Hoepli, 1919. 24mo. 20 + 353 pp. L. 6.00

MACFARLANE (A.). *Lectures on ten British physicists of the nineteenth century. (Mathematical Monographs, No. 20.)* New York, Wiley, 1919. 143 pp. \$1.50

PERNOT (F. E.). *Electrical phenomena in parallel conductors. Volume 1: Elements of transmission.* New York, Wiley, 1918. 12 + 332 pp. \$4.00

SMITH (O. S.). *Arithmetic of business.* Chicago, Lyons and Carnahan, 1917. 8vo. 6 + 448 pp. \$1.10

SOUTHALL (J. P. C.). *Mirrors, prisms and lenses.* New York, Macmillan, 1918.

STEINHEIL (A.) and VOIT (E.). *Applied optics: the computation of optical systems; edited by J. W. French from the "Handbuch der angewandten Optik."* Volume 2. London, Blackie, 1919. 213 pp. 12s. 6d.

VERHOECKX (P. M.). *De vierdimensionale wereld der relativistische natuurkunde.* Maestricht, typographie de Leiter-Nypels, 1918. 4to. 56 pp.

VOIT (E.). See STEINHEIL (A.).

Publications of the American Mathematical Society

Transactions of the American Mathematical Society.

Published quarterly. Subscription price for annual volume, \$5.00; to members of the Society, \$3.75.

Mathematical Papers of the Chicago Congress, 1893.

Price, \$3.00; to members, \$1.50.

Evanston Colloquium Lectures, 1893. By FELIX KLEIN.

Price, 75 cents.

Boston Colloquium Lectures, 1903. By H. S. WHITE, F.

S. WOODS and E. B. VAN VLECK. Price, \$2.00; to members, \$1.50.

Princeton Colloquium Lectures, 1909. By G. A. BLISS

and EDWARD KASNER. Price, \$1.50; to members, \$1.00.

Madison Colloquium Lectures, 1913. By L. E. DICKSON

and W. F. OSGOOD. Price, \$2.00; to members, \$1.50.

Cambridge Colloquium Lectures, 1916. Part I. By G.

C. EVANS. Price, \$1.50; to members, \$1.00.

**Address all orders to AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.**

Whole No. 281

CONTENTS

	PAGE
In Memory of Gabriel Marcus Green. By Professor E. J. WILCZYNSKI - - - - -	1
Reduction of the Elliptic Element to the Weierstrass Form. By Professor F. H. SAFFORD - - - - -	13
A Note on "Continuous Mathematical Induction." By Dr. Y. R. CHAO - - - - -	17
On the Number of Representations of $2n$ as a Sum of $2r$ Squares. By Professor E. T. BELL - - - - -	19
Some Functional Equations in the Theory of Relativity. By Professor A. C. LUNN - - - - -	26
Formulas for Constructing Abridged Mortality Tables for Decennial Ages. By Professor C. H. FORSYTH - - -	34
Shorter Notices - - - - -	39
Notes - - - - -	41
New Publications - - - - -	46

Changes of address of members, exchanges, and subscribers should be communicated at once to the Secretary of the American Mathematical Society, 501 West 116th Street, New York.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to The American Mathematical Society, 41 North Queen St., Lancaster, Pa., or 501 West 116th Street, New York.

The initiation fees and annual dues of members of the American Mathematical Society are payable to the Treasurer of the American Mathematical Society, Professor J. H. TANNER, Cornell University, Ithaca, N. Y.

Articles for insertion in the BULLETIN should be addressed to the
BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.

The following dates have been fixed for the meetings of the Society :

New York: October 25, 1919

San Francisco: October 25, 1919

New York: December 30-31, 1919

LIBRARY
NOV 10 1919
HARVARD COLLEGE
BULLETIN

OF THE

AMERICAN

MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

EDITED BY

F. N. COLE	VIRGIL SNYDER	J. W. YOUNG
ALEXANDER ZIWET	D. E. SMITH	
T. LEVI-CIVITA	R. C. ARCHIBALD	

VOLUME XXVI., NUMBER 2

NOVEMBER, 1919

PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK

1919

Entered at the Post Office at Lancaster, Pa., as second class matter
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER

Whole No. 282

\$5.00 a year

Digitized by Google



Introduction to the ELEMENTARY FUNCTIONS

By **RAYMOND BENEDICT McCLENON**

Edited by **WILLIAM JAMES RUSK**
Of Grinnell College

Correlates the freshman mathematics in college and introduces the student to the elements of the calculus. This unified course covers the most important topics of trigonometry, analytic geometry, and elementary differential calculus. It avoids duplication. The problems are especially well selected; the course is not too extensive for its purpose; the presentation is inductive; the style is direct and formal.

GINN AND COMPANY

70 Fifth Avenue

NEW YORK



THE TWENTY-SIXTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-sixth summer meeting of the Society was held at the University of Michigan, Ann Arbor, Mich., September 2-4, 1919, in conjunction with meetings of the Mathematical Association of America and the American Astronomical Society. At a joint dinner of the three organizations on Thursday evening Mr. J. E. Beal, of the board of regents of the University, and Professor W. W. Beman, in the absence of the president and the dean, made addresses of welcome to the visiting societies, to which Professors Schlesinger, Eisenhart and Rietz gave appropriate responses in behalf of the societies; and a formal vote of thanks for the generous hospitality of the university was adopted and subsequently presented to the proper authorities.

The Michigan Union and Newberry Residence were opened for the accommodation of the visitors, meals being served in the former building. A complimentary luncheon given by the University on Thursday, an informal reception by Professor and Mrs. Hussey at the Detroit Observatory on Tuesday evening, and auto rides around Ann Arbor and to the neighboring city of Detroit were only three of the many kindnesses of the hosts. An exhibition of the astronomical and mathematical rarities in the University of Michigan Library attracted much attention.

A meeting of the Council of the Society was held Wednesday morning in Alumni Memorial Hall. Dr. Raymond W. Brink, of the University of Minnesota, was elected to membership in the Society, and six new applications for membership were received.

At the joint dinner on Thursday evening there were one hundred and eighty persons in attendance. At the regular scientific sessions of the Society there was an attendance of over a hundred, including the following eighty members of the Society:

Dr. E. S. Allen, Professor R. C. Archibald, Professor G. N. Armstrong, Professor W. W. Beman, Professor Susan R. Benedict, Professor G. A. Bliss, Professor Henry Blumberg, Professor R. L. Borger, Professor J. W. Bradshaw, Professor

W. C. Brenke, Professor E. W. Brown, Professor W. H. Butts, Professor W. D. Cairns, Professor A. L. Candy, Professor E. H. Clarke, Dr. G. H. Cresse, Professor D. R. Curtiss, Professor S. C. Davisson, Professor F. F. Decker, Professor L. L. Dines, Professor L. W. Dowling, Professor L. P. Eisenhart, Professor John Eiesland, Professor L. C. Emmons, Professor G. C. Evans, Professor Peter Field, Professor B. F. Finkel, Professor W. B. Ford, Mr. T. C. Fry, Dr. Elizabeth B. Grennan, Professor C. F. Gummer, Professor A. G. Hall, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor L. A. Hopkins, Professor H. A. Howe, Professor E. V. Huntington, Professor W. J. Hussey, Professor L. C. Karpinski, Professor A. J. Kempner, Professor A. M. Kenyon, Professor H. R. Kingston, Professor Florence P. Lewis, Professor G. H. Ling, Professor A. C. Lunn, Dr. E. B. Lytle, Professor J. V. McKelvey, Professor W. D. MacMillan, Professor J. L. Markley, Professor G. A. Miller, Professor E. J. Moulton, Professor F. R. Moulton, Dr. A. L. Nelson, Professor F. W. Owens, Professor C. I. Palmer, Professor A. D. Pitcher, Professor L. C. Plant, Professor V. C. Poor, Professor R. G. D. Richardson, Professor H. L. Rietz, Dr. R. B. Robbins, Professor Maria M. Roberts, Professor E. D. Roe, Jr., Mrs. E. D. Roe, Professor W. H. Roever, Professor G. T. Sellew, Professor J. B. Shaw, Professor H. E. Slaughter, Professor G. W. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor R. P. Stephens, Professor E. B. Stouffer, Professor A. L. Underhill, Professor J. N. Van der Vries, Mr. H. E. Webb, Professor K. P. Williams, Professor B. F. Yanney, Professor J. W. Young, Professor Alexander Ziwet.

Thursday afternoon was devoted to a joint scientific session, at which Professor Slaughter, President of the Mathematical Association, presided. The following addresses were given:

1. "Mathematics and statistics," by Professor E. V. HUNTINGTON, being the retiring address of the former president of the Mathematical Association.

2. "The work of the National Research Council with reference to mathematics and astronomy," by Professor E. W. BROWN.

3. "Report on the international conference of scientists at Brussels," by Professor FRANK SCHLESINGER, President-elect of the American Astronomical Society.

Professor Huntington gave an introductory account of the

mathematics of statistics and urged the mathematical fraternity to give the subject more consideration than it has received in the past. His address will appear in full in the *American Mathematical Monthly* in November.

Professor Brown commented on the lack of close cooperation, previous to the period of the war, amongst the various scientific societies of our country. About a year before our entry into the war the National Research Council was organized under the enthusiastic and capable leadership of Dr. G. E. Hale, partly as a preparedness measure, with the realization that in case of war cooperation would be imperative for efficient service from our scientific men. After our entry into the war innumerable problems of a scientific character were presented to the Council and through its aid were effectively treated. Now, in time of peace, it aims to encourage research and to aid in the promotion of science and scientific interests where a broad cooperation is desirable for best results. For much of its work the Council is divided, in the tentative plans, into a number of divisions, mathematics being associated with the physical sciences group, along with astronomy, physics and geophysics. Professor Brown emphasized the fact that the National Research Council did not wish in any way to interfere with the freedom of action of any of our scientific societies, or of individual scientists, but hoped primarily to aid in matters when cooperation was obviously desirable. Concrete illustrations of possible lines of activity were suggested, including the provision, with international cooperation, of bibliographies of science and the selection of a standard system of units and notation, the propaganda for the education of people to an appreciation of the importance of science, and the securing of government aid in the development of scientific projects.

Professor Schlesinger reported the organization of and results accomplished by the International Research Council which met at Brussels, July 18-30. American astronomers took an active part in the conference but no American mathematicians were present. Professor Schlesinger gave an account of the activities of the party of American astronomers and the organization of an International Astronomical Union. An extended report of the paper will be published in *Popular Astronomy*. Dr. L. A. Bauer, of the Carnegie Institution, who also attended the Brussels conference, spoke briefly of

the organization of an International Geophysical Union as well as the International Astronomical Union.

At the regular sessions of the Society for the presentation of research papers, held Tuesday afternoon, Wednesday morning and afternoon and Thursday morning, Professors Beman, Curtiss, Snyder and Bliss served as presiding officers. Professor Karpinski served as secretary at the first session, and Professor E. J. Moulton at the other sessions. The following thirty-three papers were read:

- (1) Professor PETER FIELD: "On wind corrections."
- (2) Professor H. J. ETTLINGER: "Cauchy's memoir of 1814 on definite integrals."
- (3) Mr. L. H. RICE: "Expansion of any determinant in minors from rectangular panels."
- (4) Dr. A. L. NELSON: "Pseudo-canonical forms and invariants of systems of partial differential equations."
- (5) Professor A. J. KEMPNER: "On the separation of complex roots of an algebraic equation."
- (6) Dr. C. N. REYNOLDS, JR.: "Some theorems on the zeros of solutions of homogeneous linear differential equations of the n th order."
- (7) Dr. REYNOLDS: "Some theorems on the zeros of solutions of self-adjoint homogeneous linear differential equations of the fifth order."
- (8) Dr. J. W. ALEXANDER: "Proof of the existence of distinct three-dimensional manifolds with the same group."
- (9) Professor E. D. ROE, JR.: "Certain determinants expressible as circulants or skew-circulants."
- (10) Professor W. B. FORD: "A brief account of the life and work of the late Professor Ulisse Dini."
- (11) Dr. S. P. SHUGERT: "The resolvents of König and other types of symmetric functions."
- (12) Professor G. A. MILLER: "Form of the number of subgroups of prime-power groups."
- (13) Dr. E. S. ALLEN: "A generalization of a formula of Schubert in enumerative geometry."
- (14) Dr. E. P. LANE: "Joint-axis congruences with indeterminate developables."
- (15) Dr. R. W. BRINK: "A modification of an integral test for the convergence and divergence of infinite series."
- (16) Professors F. R. SHARPE and VIRGIL SNYDER: "Certain types of involutorial space transformations (second paper)."

(17) Professor L. P. EISENHART: "Transformations of surfaces applicable to a quadric."

(18) Professor EISENHART: "Transformations of cyclic systems of circles."

(19) Professor G. A. BLISS: "Differential equations containing arbitrary functions."

(20) Professor BLISS: "Functions of lines in ballistics."

(21) Professor D. R. CURTISS: "On the relative positions of the complex roots of an algebraic equation with real coefficients and those of its derived equation."

(22) Professor H. L. RIETZ: "Urn schemata as a basis for the development of the theory of correlation."

(23) Professor L. L. DINES: "Projective transformations in function space (second paper)."

(24) Professor J. B. SHAW: "On the invariants belonging to a hypernumber in an algebra of infinite order."

(25) Professor R. D. CARMICHAEL: "Conditions necessary and sufficient for the existence of a Stieltjes integral."

(26) Professor CARMICHAEL: "Note on convergence tests for series and on Stieltjes integration by parts."

(27) Professor CARMICHAEL: "Note on a physical interpretation of Stieltjes integrals."

(28) Professor H. A. HOWE: "An apparent anomaly in errors of interpolated values."

(29) Professor G. C. EVANS: "Transformations of a Stieltjes integral potential."

(30) Professor T. H. HILDEBRANDT: "Note on sequences of Stieltjes integrals."

(31) Professor W. H. ROEVER: "Equations of motion of a projectile regarded as a particle."

(32) Professor W. D. CAIRNS: "Certain properties of binomial coefficients."

(33) Professor A. J. KEMPNER: "On the shape of polynomial curves."

In the absence of the authors the papers by Professor Ettlinger, Mr. Rice, Dr. Reynolds, Dr. Alexander, Dr. Shugert, Dr. Lane, Dr. Brink, and Professor Carmichael, were read by title only. The papers on ballistics led to a somewhat extended discussion by Professors Field, Bliss, Roever, and F. R. Moulton. Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles above:

1. Professor Field's paper deals with the fundamental assumptions on which the ordinary wind correction formulas are based. It will be published in the *Journal of the U. S. Artillery*.

2. In this paper Professor Ettlinger discusses Cauchy's memoir of 1814 from a historical and critical point of view. Eminent mathematicians of that day, such as Poisson, Laplace and Legendre, did not properly appreciate the importance of the contribution and merely saw in the memoir new methods of evaluating definite integrals which Euler had computed years before. For the modern reader, also, it is difficult to find in this memoir the very first statement of Cauchy's discoveries on residues and the integral theorem upon which the structure of the modern theory of functions of a complex variable is built. These facts the author brings into the light by simplifications, both algebraic and geometrical. The method is also made to conform to our modern standard of rigor.

The subject of this paper was suggested by Professor W. F. Osgood.

3. By availing himself of the full force of a new proof of Muir's recent theorem (*Messenger of Mathematics*, May, 1918) that "a determinant can be expressed in terms of minors drawn from four mutually exclusive arrays, two of which are coaxial and complementary to one another," Mr. Rice shows, not only that the last clause of the theorem can be freed of the restriction to coaxial (square) arrays, but also that there need not be precisely four rectangular arrays or panels but the matrix of the determinant may be subdivided into any number of panels independent of one another in point of dimensions, complementary sets of minors being drawn from these panels.

4. In a recent note (*American Journal of Mathematics*, volume 41, April, 1919, pages 123-132), Dr. Nelson has shown that fundamental sets of seminvariants of a completely integrable system of linear homogeneous partial differential equations may be found by the reduction of the system to certain pseudo-canonical forms. The method is here extended to include semi-covariants, invariants and covariants. Simul-

taneous invariants of several such completely integrable systems may be similarly constructed. Application is also made to semi-covariants of algebraic forms.

5. By simple vector considerations Professor Kempner determines certain sectors starting at the origin in the complex plane which do not contain roots of a given algebraic equation. These sectors depend, in case the equation has real coefficients, only on the sequence of signs of the coefficients; in case the equation has complex coefficients, only on the arguments of the coefficients. When the equation is of low degree or when, while the degree of the equation is high, many coefficients vanish, the information obtained is often sufficient to separate entirely the complex roots. In most cases the information is less precise. The method applies also to equations containing fractional powers of the unknown.

6. In this paper Dr. Reynolds proves a general separation theorem for the zeros of solutions of homogeneous linear differential equations of the n th order and compares the zeros of the solutions of the two differential systems

$$\begin{aligned} (1) \quad & y^{(n)}(x) + \sum_{i=2}^n p_i(x)y^{(n-i)}(x) = 0, \\ & y^{(k)}(\xi) = \alpha_k \quad (k = 0, 1, 2, 3, \dots, n-1), \\ (2) \quad & \eta^{(n)}(x) + \sum_{i=2}^{n-1} p_i(x)\eta^{(n-i)}(x) + [p_n(x) + R(x)]\eta(x) = 0, \\ & \eta^{(k)}(\xi) = \alpha_k \quad (k = 0, 1, 2, 3, \dots, n-1). \end{aligned}$$

For n even the theorems obtained generalize the author's earlier results concerning equations of the fourth order. For n odd the theorems obtained generalize Professor Birkhoff's results concerning equations of the third order (*Annals of Mathematics*, second series, volume 12).

7. Dr. Reynolds applies the general separation theorem of the preceding paper to the solutions of self-adjoint homogeneous linear differential equations of the fifth order. Any solution of such an equation can be expressed linearly in terms of the six functions $\eta_i\eta_j' - \eta_j\eta_i'$ ($i, j = 1, 2, 3, 4; i < j$), where the η 's satisfy a homogeneous self-adjoint linear differential

equation of the fourth order. The wronskian of any two solutions of the fifth order equation is a homogeneous quadratic form in the η 's. By means of these two relations the author proves a group of theorems concerning the solutions of self-adjoint equations of the fifth order which are of the same degree of generality as his earlier results concerning self-adjoint equations of the fourth order and interprets his results geometrically by means of the integral curve of the self-adjoint equation of the fourth order which is satisfied by the η 's.

8. In this paper, Dr. Alexander shows that two distinct three-dimensional manifolds with the same Betti numbers, coefficients of torsion, and group exist. The Heegaard diagrams of the manifolds consist of anchor rings on which have been traced the curves ab^5 and a^2b^5 respectively.

9. Generalizing, Noether and Puchta discovered determinants of m elements, and of order m , which were factorable into m linear factors of their elements, the factors being real for $m = 2^n$. Noether showed that, for $m = n_1 n_2 \cdots n_k$, the determinant is the simultaneous resultant of a first set of equations $x_1^{n_1} - 1 = 0, x_2^{n_2} - 1 = 0, \dots, x_k^{n_k} - 1 = 0$, and $\Sigma a_{i_1 i_2 \dots i_k} x_1^{i_1} x_2^{i_2} \dots x_k^{i_k} = 0$, where the degree of Σ in $x_1, x_2, \dots, x_k$ is $n_1 - 1, n_2 - 1, \dots, n_k - 1$ respectively. Professor Roe extends the generalization to the resultant of a second set of equations $x_1^{n_1} + 1 = 0, x_2^{n_2} + 1 = 0, \dots, x_k^{n_k} + 1 = 0$, and $\Sigma = 0$. He points out that by eliminating one variable x_j by Sylvester's method a circulant or a skew-circulant respectively, in the remaining variables, of order n_j results. The elimination of a second variable, not by Sylvester's method, but by forming the product of the function obtained by substituting all the values of that variable, obtained from its equation in the first or second set respectively, in the circulant or skew-circulant already formed, yields, by the laws for the multiplication of circulants or skew-circulants, a circulant or skew-circulant of order n_j , and containing one less variable. Continuation of the elimination in this way, step by step, gives the resultant finally as a circulant or skew-circulant of order n_j . As the variable x_j first eliminated was chosen at pleasure, it follows that the determinants of Noether and Puchta and their analogues

here suggested by Professor Roe can be expressed as circulants or skew-circulants respectively of any of the orders $n_1, n_2, \dots, n_k$. Numerous identities follow from these different expressions. As a simple example of such identities, let $k = 2$, $n_1 = 2$, $n_2 = 3$. Then from one or the other form we can derive $A^3 + B^3 \equiv a^2 - b^2$, $A'^3 - B'^3 \equiv a'^2 + b'^2$, where $A, B, A', B', a, b, a', b'$ are functions of 5 independent of 6 elements, another function C or C' being placed equal to zero to secure the form. The identities are satisfied in ∞^5 ways. They can be generalized to such as are satisfied in $\infty^{2n-1} = \infty^{2p-1}$ ways by $2n - 1$ independent elements, where $n \equiv 0 \pmod{3}$, $p \equiv 0 \pmod{2}$, and these in turn can be still further generalized. Professor Roe also considers the resultant of the elimination between a third set of equations $x_1^{n_1} - 1 = 0, \dots, x_k^{n_k} - 1 = 0$, $x^{m_1} + 1 = 0, \dots, x^{m_s} + 1 = 0$ and $\Sigma = 0$, by virtue of which the resultant can be shown to be a circulant of any of the orders $n_1, \dots, n_k$ or a skew-circulant of any of the orders $m_1, \dots, m_s$. He also considers the resultant between any of the systems and $\Sigma = 0$, when all restrictions as to the degree of Σ are removed, and shows that according as $d_j \geq n_j - 1$ (d_j the degree of x_j in Σ), circulants or skew-circulants with polynomial or zero elements result. Also such identities as $A^2 + B^2 \equiv A'^2 - B'^2$, $A^3 + B^3 \equiv A'^3 - B'^3$, etc.

10. Professor Ford's paper is intended to supplement the brief announcement of Professor Dini's death as recorded in the BULLETIN of last March and April.

11. Dr. Shugert's paper treats the following subjects, in part by methods and formulas derived or suggested by Professor Glenn in a paper on the theory of finite expansions in volume 15 of the *Transactions*: a lemma on determinants whose elements above the diagonal next above the principal diagonal are all zero; the expansion of a polynomial in ascending powers of a quadratic polynomial; conditions that a form contain a given quadratic form as a factor; separation of $a_x^m/(\xi_x^2)^k$ into partial fractions; the explicit formation, by determinants, of the resolvents studied by König in the *Mathematische Annalen*, volume 15 (1879); tables of symmetric functions of the sums and of the products, in pairs, of the roots of a given binary form. These tables are complete for the forms of orders 2 to 6 inclusive.

The paper is a University of Pennsylvania doctor dissertation and is being printed by the New Era Press by private arrangement with the author.

12. Professor Miller's paper appears in full in the present number of the BULLETIN.

13. If, on an algebraic curve of genus p , an algebraic series of point groups γ_m^1 of dimension 1 and a linear series g_n^r of any dimension r are given, the formula of Schubert states the number of those groups of γ_m^1 which contain $r+1$ points of a single group of g_n^r .

Dr. Allen obtains the number of those groups of a series γ_m^2 , of dimension 2, which contain $r+2$ points of a single group of a given g_n^r . He obtains a recurrence formula by means of two pairs of correspondences between points of the curve, and from this finds the closed formula by mathematical induction.

The formula has the following two simple forms, which involve, respectively, the characters z_b of the series γ_m^2 , and the numbers $d_{1[a]}$, of groups of γ_m^2 possessing a double points, together with enough arbitrary fixed points to make $d_{1[a]}$ result finite:

$$\begin{aligned} Z_{n, r, m, 2} &= \sum_{b=0}^2 \sum_{h=b}^2 (-1)^{h+b} \binom{m-2-h}{r-h} \binom{n-r-h}{2-h} \binom{p-b}{h-b} z_b \\ &= \sum_{a=0}^2 \sum_{h=a}^2 (-2)^{-a} \binom{m-2-a}{r-a} \binom{r-a}{h-a} \binom{n-r-h}{2-h} d_{1[a]}. \end{aligned}$$

14. The joint-axis of a point P_v on a surface S_v , referred to a conjugate net, and of the corresponding point P_s on the first Laplace transform S_s of the net, has been defined by Wilczynski to be the line of intersection of the osculating plane of the curve $u = \text{const.}$ at P_v and the osculating plane of the curve $v = \text{const.}$ at P_s . Thus, all the joint-axes of pairs of corresponding points constitute a congruence. Dr. Lane studies the properties of the joint-axis congruence, and of the suite of Laplace transforms of S_v , which are characteristic of this configuration when the developables of the joint-axis congruence are indeterminate. He finds, by the methods of projective differential geometry, using the nota-

tion of Wilczynski's system (D), that there are two types of such configurations. In the first type, the joint-axes all pass through a fixed point, to which the third and minus second Laplace transforms of S_v reduce. In the second type, the joint-axes all lie in a fixed plane and give rise to a planar net of period three. The plane is characterized geometrically, and a number of geometrical theorems are obtained which relate certain points and lines in the configuration which have escaped notice hitherto.

15. In a previous paper Dr. Brink presented an integral test for the convergence and divergence of infinite series. It is applicable to series for which a simple analytic expression for a function $r(x)$ is known, where $r(n)$ is the ratio of the n th term of the series to the preceding term. And in this test $r(x)$ plays a rôle analogous to that played in the familiar Maclaurin-Cauchy integral test by the function $u(x)$, where $u(n)$ is the n th term of the series. In the present paper is given a modification of this test, in which a new form of the integral is introduced. This permits of a certain simplification of the conditions of the theorem, and a great simplification in the demonstration.

16. The paper of Professors Sharpe and Snyder is supplementary to that having the same title in the *July Transactions*. A general method of determining (1, 2) correspondences is developed and is applied to discuss the possible cases in which the order of the surfaces that are mapped on the planes of the double space does not exceed five. The discussion includes among others those involutions in which the surface of invariant points is birationally equivalent to a quartic with ten or more nodes.

17. In volume 19 of the *Transactions* Professor Eisenhart established a transformation of a pair of applicable surfaces into new pairs of applicable surfaces. In the present paper he shows that the transformations of surfaces applicable to a central quadric determined at great length by Guichard in the memoir which received the grand prize of the French academy in 1909 are of this kind, and that they follow readily from the general theory. Similar transformations of surfaces applicable to a paraboloid are established. The known trans-

formations of surfaces of constant curvature are likewise of this type. The general transformations of surfaces applicable to the quadrics admit two theorems of permutability, in consequence of which from known transformations others follow directly.

18. When the circles of a two-parameter family of circles are orthogonal to a one-parameter family of surfaces Σ , the circles are said to form a cyclic system. In the correspondence established between the surfaces by the circles, the lines of curvature correspond on all the surfaces, and to them correspond also the developables of the congruence formed by the axes of the circles. The planes of the circles envelope a surface S upon which the curves corresponding to the lines of curvature on the surfaces Σ form a conjugate net N , which admits an applicable conjugate net $\bar{N}$. Professor Eisenhart applies to these applicable nets the theorem referred to in his former paper and establishes thereby a transformation of the given cyclic system into other cyclic systems. Corresponding circles of a cyclic system and a transform lie on a sphere, and the nets N and N_1 on the surfaces S and S_1 enveloped by the planes of the circles are in relation T , that is, the developables of the congruence of joins of corresponding points on S and S_1 meet the latter in N and N_1 . Moreover, it is possible to establish a correspondence between the surfaces Σ and Σ_1 , orthogonal to the two families of circles, so that a surface Σ and the corresponding surface Σ_1 are in the relation of a transformation of Ribaucour, that is, they are the envelope of a two-parameter family of spheres with lines of curvature in correspondence on the two sheets of the envelope.

19. If in a set of differential equations

$$\frac{dx_i}{d\tau} = f_i(\tau, x_1, \dots, x_n) \quad (i = 1, \dots, n)$$

the functions in the second members are considered as variables, the solutions x_i ($i = 1, \dots, n$) will be functions, not only of the initial values of τ and the x 's, but also of the functions f_i themselves. In Professor Bliss's first paper the properties of these functions of functions are discussed. It is proved, after suitable initial hypotheses have been made, that they are continuous, and that they have what are called

in this paper difference functions. It is shown that the possession of a difference function implies also the possession of a differential similar to those which have been defined by Fréchet and Volterra for functions of lines. The theorems of the paper are widely applicable since the variations Δf_i of the functions f_i entering into the differentials are entirely arbitrary. In special cases they may be caused, for example, by variations of parameters in the functions f_i , or, as in ballistics, by variations of other arbitrary functions which occur as arguments in the second members of the differential equations.

20. In Professor Bliss's second paper some functions of lines are considered which arise in ballistics. If $w(y)$ is the velocity of the wind at the altitude y in the direction of the plane of fire of a projectile, then the range X will be a function of the function $w(y)$, and also of the projections x_0' , y_0' on the axes of the initial velocity of the projectile. The first differential of this function $X[x_0', y_0', w(y)]$ is the first order effect on the range of wind and of changes in the direction and magnitude of the initial velocity. The paper gives a method of computing the first differential which is of interest theoretically in the theory of functions of lines, and which has proved to be of value in the practical computation of range tables. The situation just described is somewhat simpler than that which arises in practice, for the range is also affected by variations from normal in the density of the air and by the rotation of the earth, and still other disturbing influences will have to be accounted for in the future as the applications of ballistic theory become more exacting. The formulas of the paper are applicable to the cases which have just been mentioned explicitly, and they will undoubtedly also be useful for the computation of other differential corrections which may be necessary in the future. The results obtained are special cases of those of the preceding paper, but they are here deduced independently except for some general theorems which would very likely be accepted by an applied mathematician as intuitively true.

21. Jensen has recently stated, without proof, a theorem regarding the roots of the equations $f'(z) = 0$ and those of $f(z) = 0$, where the coefficients are all real, which locates the

former in many cases more closely than does the Gauss-Lucas "polygon theorem." Professor Curtiss's paper gives a simple proof of Jensen's theorem, and deduces from it the result that at least one pair of roots of $f(z) = 0$ lies in each rectangular hyperbola whose vertices are a conjugate imaginary pair of roots of $f'(z) = 0$. This result is extended so as to give similar theorems regarding the relative positions of roots of $f(z) = 0$ and its derived equations of all orders.

22. It is well known that simplicity and precision are gained by the use of urn schemata in establishing various theorems in the theory of probability. The fundamental importance of urn schemata in mathematical statistics is brought out well by Borel in his *Eléments de la Théorie des Probabilités* by the statement that the general problem of mathematical statistics is to determine a system of drawings carried out with urns of fixed composition, in such a way that the results of the series of drawings, interpreted by the use of conveniently chosen coefficients, lead, with a very high degree of probability, to a table of values identical with the table of observed values. In the present paper, Professor Rietz presents several urn schemata to show the significance of the correlation coefficient derived from a priori probabilities or most probable frequencies given by pure games of chance. It turns out that in certain cases the correlation coefficient r is simply the ratio of the number of common elements to the total number involved. In other cases, the results, though not so simple, are still capable of interesting interpretations. It is emphasized that such urn schemata may well be made the starting point in the development of the theory of correlation.

23. The theory of the transformations studied by Professor Dines in a paper of the same title in the *Transactions* (volume 20, January, 1919) is further developed in the present paper. The introduction of homogeneous coordinates is attended by the usual advantages. A class of points satisfying a certain type of linear equation is called a lineoid, and serves as an element dual to the point. The question of invariant elements is treated, and provides a means for classifying projective transformations. A theorem of special interest states: A necessary and sufficient condition that a projective trans-

formation be projectively equivalent to an ordinary Fredholm transformation in the same domain is that it admit an invariant lineoid and an invariant point not on that lineoid. In particular every symmetrical projective transformation has this property, but such is not the case with non-symmetrical transformations.

24. The invariants referred to in Professor Shaw's paper are the scalar invariants which appear in the characteristic equation, the expressions previously called *chi*, and related forms. The subject is intimately connected with the Fredholm theory in integral equations.

25. The purpose of the first paper by Professor Carmichael is to suggest a method for deriving a necessary and sufficient condition for the existence of the Stieltjes integral of $f(x)$ as to $u(x)$ from a to b in each of several forms generalizing those frequently employed in the special case of Cauchy-Riemann integration. The theorem of Bliss (*Proceedings of the National Academy of Sciences*, volume 3 (1917), pages 633-637) is included and is proved in a new way.

26. An identity to which he was led by the problem of Stieltjes integration by parts Professor Carmichael applies in the derivation of several theorems relating to the convergence of series. Of these the following is typical: The convergence of the series

$$\sum_{k=1}^{\infty} c_{k-1}^{(2)} c_{k-1}^{(3)} \dots c_{k-1}^{(n)} (c_k^{(1)} - c_{k-1}^{(1)})$$

is implied by the convergence of the series

$$\sum_{k=1}^{\infty} |c_k^{(i)} - c_{k-1}^{(i)}|, \quad (i = 2, 3, \dots, n),$$

and the existence of the limits

$$\lim_{k \rightarrow \infty} c_k^{(1)} c_k^{(2)} \dots c_k^{(n-1)}, \quad {}^* \lim_{k \rightarrow \infty} c_k^{(1)} \dots c_k^{(i-1)} c_{k-1}^{(i+1)} \dots c_{k-1}^{(n)},$$

($i = 2, \dots, n-1$).

The instance $n = 2$ of this theorem is classic.

27. In this paper Professor Carmichael throws Stieltjes'

problem of moments into a different form from that of Stieltjes and one readily capable of generalization; and from this he proceeds to a natural physical interpretation of the Stieltjes integral of a bounded function $f(x)$ as to a function $v(x)$ of bounded variation.

28. On page 506 of Merriman and Woodward's Higher Mathematics is given a table showing—for different values of the interpolating factor—the actual and the theoretical average errors of certain interpolated values. Attention is called to the curious fact that when the interpolating factor is an even number the actual error is always less than the theoretical error, while when it is odd the reverse is true, except when the factor is 0.5.

In Professor Howe's paper it is purposed to describe a more exhaustive test of this matter, with special attention to the case where the interpolating factor is 0.5, and also to call attention to an Austrian five-place table from which more accurate results can be obtained than from ordinary tables. Also when only the usual accuracy is sought one can work much more rapidly with this table than with others.

29. In the current volumes of the *Rendiconti* of the Lincei and of the *Seminario* of the University of Rome, Professor Evans expounded a theory of the general equation of Poisson,

$$\int \frac{\partial u}{\partial n} ds = f(e),$$

where $f(e)$ is an arbitrary additive function of point sets, in which he introduced a potential function in the form of a Stieltjes integral. The present report discusses a transformation of this integral into the form involving a polarization vector, and a second form obtained by the slight generalization of an integration by parts of W. H. Young. In the latter case are introduced nuclei which in three dimensions attract according to an inverse fifth power law.

30. In this note Professor Hildebrandt derives a necessary and sufficient condition in order that a sequence of Stieltjes integrals $\int f d\alpha_n$ shall converge to an integral $\int f d\alpha$, for every continuous function on the linear interval $a \leq x \leq b$, the functions α_n and α being of bounded variation on the same

interval. He also shows that a linear functional operation on the class of functions of bounded variation on the interval (a, b) is expressible in the form of a Stieltjes integral $\int f d\alpha$, in which f is a continuous function on (a, b) ; and derives necessary and sufficient conditions that the sequence of Stieltjes integrals $\int f_n d\alpha$ shall converge to an integral of the form $\int f d\alpha$, for every function of bounded variation on the interval (a, b) , the functions f_n and f being continuous on the same interval.

31. Professor Roevers's paper begins with a statement of the theorems of relative motion concerning the velocities and accelerations of a particle. By applying these theorems to the problem under consideration the equation of motion of a projectile is expressed in vector form in terms of the accelerations due to weight, wind, air resistance and rotation of the earth. From this vector equation the equations of motion are obtained in a form involving the potential function W of the weight field of force, the function F which expresses the retardation due to air resistance and wind, and the angular velocity of the earth's rotation. By replacing W by means of some of its analytic approximations and expressing F in terms of various experimentally determined functions, differential equations of motion of a projectile are obtained which take into consideration air resistance, wind, rotation of the earth, convergence of the verticals, curvature of the layers of constant air density and also other influences, but not the rotation of the projectile due to rifling of the gun.

32. There are known summations of products of the successive coefficients of $(x - y)^k$ by given powers of the numbers of those coefficients, but no results seem to be known for the case of the binomial coefficients, i.e., those of $(x + y)^k$. A recursion formula is given by Professor Cairns for such summations as well as expressions for the leading terms, and an application is made to the evaluation of

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} x^{2p} dx \quad \text{and} \quad \int_0^{\infty} e^{-x^2/2\sigma^2} x^{2p+1} dx.$$

33. Professor Kempner gives a simple method of classification of real polynomial curves with reference to the relative

position and the number of the maxima and minima of the curve, and shows that all types actually exist. For example: for all $\delta > 0$ sufficiently small, the curve

$$y = (x + \delta i)(x - \delta i)(x + 1 + \delta^3 i)(x + 1 - \delta^3 i) \cdot \\ (x + 1 + \delta^2 + \delta^5 i)(x + 1 + \delta^2 - \delta^5 i)(x + 1 + \delta^2 + \delta^4)$$

has six extremes which, read for decreasing values of x , are arranged so that the first minimum of y is higher than the second maximum, and the second minimum higher than the third maximum.

E. J. MOULTON,
Acting Secretary.

FORM OF THE NUMBER OF SUBGROUPS OF PRIME POWER GROUPS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society September 3, 1919.)

§1. *Introduction.*

It is known that the number of the subgroups of order p^a , p being any prime number, which are contained in any group G is always of the form $1 + kp$. When $k = 0$ for every possible pair of values for α and p the group G must be cyclic and vice versa. There are two infinite systems of groups of order p^m containing separately $p + 1$ subgroups of every order which is a proper divisor of the order of the group, viz., the abelian groups of type $(m - 1, 1)$ and the conformal non-abelian groups.

These two infinite systems are composed of all the groups of order p^m involving separately exactly $p + 1$ subgroups of every order which is a proper divisor of p^m . Moreover, if a group of order p^m , $p > 2$, contains exactly $p + 1$ subgroups of each of the two orders p and p^2 it must contain exactly $p + 1$ subgroups of every order which is a proper divisor of the order of the group, and if a group of order 2^m contains exactly three subgroups of each of the orders 2, 4 and 8 it must also contain exactly three subgroups of every other order which is a proper divisor of 2^m .

The proof of the former part of this theorem may be based upon the fact that when $m > 3$ such a group has to contain an invariant abelian subgroup of order p^3 and of type $(2, 1)$. The order of each of the remaining operators must be divisible by p^3 and the number of the operators of order p^3 is equal to $p^4 - p^3$. Hence G contains exactly p cyclic subgroups of order p^3 and only one non-cyclic subgroup of this order. If $m > 4$, this subgroup of order p^4 must again be abelian, and by successive similar steps we note that G must contain operators of order p^{m-1} .

When $p = 2$ it is possible to extend the abelian group of order 8 and of type $(2, 1)$ by 24 operators of order 4 so as to obtain a group of order 32. In this case the preceding reasoning fails. If it is assumed, however, that G contains exactly three subgroups of order 8 in addition to the three subgroups of each of the orders 2 and 4 it may be proved as in the preceding paragraph that G contains only $p = 2$ cyclic subgroups of order 16, etc. The theorem stated in the second paragraph has therefore been established. This theorem may be compared with the well known theorem, due to W. Burnside, which affirms that every group of order p^m , $p > 2$, which contains only one subgroup whose order is a proper divisor of p^m is cyclic.

Another simple condition which implies that a group G of order p^m is cyclic is that all the operators of G which transform into itself one of its subgroups constitute a cyclic subgroup of G . In other words, a necessary and sufficient condition that a prime power group is cyclic is that it contains at least one subgroup whose normaliser is cyclic. This theorem results directly from the fact that every non-invariant subgroup of a group of order p^m is transformed into itself by at least p of its conjugates including itself. In the following section it will be proved that a necessary and sufficient condition that an abelian group of order p^m is cyclic is that the number of its subgroups of order p^α , $0 < \alpha < m$, is of the form $1 + kp^2$ for at least one value of α .

§2. Abelian Groups.

Let H represent a subgroup of order p^α contained in an abelian group G of order p^m and suppose that H has been so selected that its invariants are as small as possible when the

order of H is fixed. It may happen that H is composed of all the operators of G whose orders do not exceed the largest invariant of H . If this is the case, G contains only one subgroup of order p^α having the same invariants as H . If it is not the case, we proceed to prove that the number of subgroups of G which have the same invariants as H must be of the form $1 + p + kp^2$.

Let $p^{\alpha+\lambda}$ represent the order of the subgroup of G composed of all its operators whose orders divide the largest invariant of H . If H has β such largest invariants, the β independent generators of H whose orders are equal to these invariants can be selected from the operators of G in a number of ways represented by the following product:

$$(p^{\alpha+\lambda} - p^{\alpha-\beta})(p^{\alpha+\lambda} - p^{\alpha-\beta+1}) \dots (p^{\alpha+\lambda} - p^{\alpha-1}).$$

Similarly, these β independent generators can be selected from the operators of H in

$$(p^\alpha - p^{\alpha-\beta})(p^\alpha - p^{\alpha-\beta+1}) \dots (p^\alpha - p^{\alpha-1})$$

different ways, and the remaining independent generators can be selected in the same number of ways from the operators of G and those of H . When $\lambda > 0$, the quotient of the first of these two products divided by the second is evidently of the form $1 + p + kp^2$, and hence the following theorem has been established:

THEOREM. *The number of the subgroups of order p^α contained in an abelian group of order p^m and satisfying the condition that their invariants are as small as possible when α is given is either unity or of the form $1 + p$, mod p^2 .*

As a special case of this theorem it may be noted that the number of subgroups of order p^α contained in an abelian group of order p^m , $m > \alpha > 0$, and of type $(1, 1, 1, \dots)$ is always of the form $1 + p$, mod p^2 . In this special case α and β are always equal to each other.

Suppose that G contains only one subgroup H of order p^α such that its invariants are as small as possible and consider the subgroups of order p^α which have the property that their largest invariant is p times a largest invariant of H and that a second invariant is equal to one of the largest remaining invariants of H divided by p while the rest of the invariants of such a subgroup H' are the same as those of H . We shall

prove that the number of the subgroups of G satisfying the conditions imposed on H' is always of the form $p + kp^2$, so that in this case the number of subgroups of G which satisfy the conditions imposed on H and H' is $1 + p \bmod p^2$.

It will first be assumed that H contains $\beta > 2$ largest invariants and that the subgroup of G composed of all its operators whose orders divide this largest invariant multiplied by p is of order $p^{\alpha+\lambda}$. Hence $\lambda \leq \beta \leq \alpha$. The number of ways in which a set of $\beta - 1$ largest independent generators of H' can be selected from the operators of G is expressed by the following product:

$$(p^{\alpha+\lambda} - p^{\alpha})(p^{\alpha} - p^{\alpha-\beta+1}) \dots (p^{\alpha} - p^{\alpha-2}).$$

The number of ways in which these generators can be selected from the operators of H' is represented by the following product:

$$(p^{\alpha} - p^{\alpha-1})(p^{\alpha-1} - p^{\alpha-\beta+1}) \dots (p^{\alpha-1} - p^{\alpha-2}).$$

As the remaining independent generators of H' can be selected in the same number of ways from the operators of G and from those of H' , the quotient of the given products is equal to the number of subgroups of G satisfying the conditions imposed on H' . This number is evidently of the form $p + kp^2$, and hence the theorem in question has been proved whenever $\beta > 2$.

When $\beta = 2$ the largest independent generator of H' can be selected from the operators of G and from those of H' in

$$p^{\alpha+\lambda} - p^{\alpha} \text{ and } p^{\alpha} - p^{\alpha-1}$$

ways respectively while the remaining independent generators of H' can be selected in the same number of ways from each of these two groups. Hence the number of subgroups satisfying the conditions imposed on H' is $p(p^{\lambda-1} + p^{\lambda-2} + \dots + 1)$ in this case.

Finally, when $\beta = 1$ and H' contains $\gamma > 1$ next to the largest invariants, the γ largest independent generators of H' can be selected from the operators of G in

$$(p^{\alpha+1} - p^{\alpha})(p^{\delta} - p^{\delta-\gamma}) \dots (p^{\delta} - p^{\delta-2})$$

different ways, p^{δ} being the order of the subgroup of G composed of all its operators whose orders divide the second

largest invariant of H' . These independent generators can be selected from the operators of H' in

$$(p^a - p^{a-1})(p^{b-1} - p^{b-\gamma}) \dots (p^{b-1} - p^{b-2})$$

different ways and hence the number of these subgroups is again of the form $p + kp^2$. This completes a proof of the following theorem, since the case $\gamma = 1$ is evidently included therein:

THEOREM. *If a subgroup H of an abelian group G of order p^m is of order p^a and composed of all the operators of G whose orders divide a given number, then the number of the subgroups of G whose largest invariant is p times the largest invariant of H and whose second invariant is obtained by dividing by p the largest of the remaining invariants of H , while its other invariants are the same as the rest of the invariants of H , is always of the form $p + kp^2$.*

The preceding theorems have been established with a view to proving that the number of proper subgroups of order p^a contained in a non-cyclic abelian group G of order p^m is always of the form $1 + p \bmod p^2$. To complete the proof of this theorem it is only necessary to establish the fact that the number of subgroups of order p^a having a different type from those considered above must always be of the form kp^2 , k being a natural number. As a step in the proof of this theorem we note the following fundamental fact which entered the preceding considerations in a special form.

An independent generator of order p^b of the subgroup H of order p^a can be selected, if these generators are selected in the descending order of magnitude and k of them have already been selected, from the operators of G in

$$p^r - p^{s+k}$$

different ways, where p^r and p^s represent the orders of the subgroups of G composed of all its operators whose orders divide p^b and p^{b-1} respectively. Similarly, this independent generator can be selected from the operators of H in

$$p^{r'} - p^{s'+k}$$

different ways, where $r' \leq r$ and $s' \leq s$. To prove the theorem in question it is therefore only necessary to prove that either some s is at least two units larger than the corresponding

s' or at least two s 's are each one unit larger than the corresponding s 's.

From this fact it follows that the highest power of p which divides the number of the subgroups of order p^a and of type $(\alpha_1, \alpha_2, \dots, \alpha_\lambda)$ may be found as follows: Determine the orders of the characteristic subgroups composed separately of all the operators whose orders divide $p^{\alpha_1-1}, p^{\alpha_2-1}, \dots, p^{\alpha_\lambda-1}$ in G and in a particular subgroup H of type $(\alpha_1, \alpha_2, \dots, \alpha_\lambda)$ respectively. The product of the orders of these subgroups of G divided by the product of the orders of the corresponding subgroups of H gives a quotient which is the power of p in question. For instance, if G is of type $(6, 6, 5, 4, 2)$ and H is of type $(6, 3, 2, 1)$ the number of subgroups of G which are of order p^{12} and of type $(6, 3, 2, 1)$ is divisible by

$$p^{21+10+5-11-7-4}$$

but not by any higher power of p .

In particular, it may be noted that the number of subgroups of order p^a and of a given type is always divisible by p^2 whenever the number of the invariants of G exceeds the number of the invariants of such a subgroup H by 2 and at least one of the latter invariants exceeds p . If at least two of these invariants exceed p , the number of these subgroups is divisible by p^2 whenever G has at least one more invariant than H has. Hence when the number of the subgroups of the same type as H has is not divisible by p^2 the number of invariants of H is either the same as that of G or one less than that of G except when H is of type $(1, 1, 1, \dots)$.

Moreover, when H has one invariant less than G and the number of subgroups having the same type as H has is not divisible by p^2 , it results from the preceding considerations that either no invariant of H exceeds p , or that only one of these invariants exceeds p . In the latter case this invariant is p^2 unless G has also only one invariant greater than p . Hence the following:

THEOREM: *Whenever the number of subgroups of the same type as H is not divisible by p^2 and the number of invariants of H is less than the number of invariants of G there is one and only one subgroup in G having the same order as H but smaller invariants than H has.*

It remains only to consider the case when the subgroups of order p^a which have the same invariants as H have as

many invariants as G . Suppose that the invariants of G arranged in descending order of magnitude are $p^{\alpha_1}, p^{\alpha_2}, \dots, p^{\alpha_\lambda}$ while those of H arranged similarly are $p^{\alpha'_1}, p^{\alpha'_2}, \dots, p^{\alpha'_\lambda}$. It is well known that $\alpha_1 \geq \alpha'_1, \alpha_2 \geq \alpha'_2, \dots, \alpha_\lambda \geq \alpha'_\lambda$. The number of the subgroups of type $(\alpha'_1, \alpha'_2, \dots, \alpha'_\lambda)$ is evidently divisible by p^2 whenever α'_1 is at least two units larger than each of two other α' 's which are separately smaller than the corresponding α 's, and also when α'_1 and α'_2 are separately two units larger than some one α' which is less than the corresponding α , or α'_1 is at least three units larger than such an α' provided this α' is not α'_2 and $\alpha_2 = \alpha'_2 + 1$.

From the preceding paragraph it results that whenever the number of subgroups of type $(\alpha'_1, \alpha'_2, \dots, \alpha'_\lambda)$ is not divisible by p^2 and an α'_β is less than α_β then there is at most one other α' which is two units larger than α'_β . If there is such an α' it is α'_1 and $\alpha'_\gamma, \beta > \gamma > 1$, is equal to $\alpha'_\beta + 1$. It was noted above that when these conditions are satisfied the number of subgroups of G which are of type $(\alpha'_1, \alpha'_2, \dots, \alpha'_\lambda)$ is of the form $p + kp^2$. This is also the case when $\alpha'_\lambda = \alpha_\lambda, \alpha'_{\lambda-1} = \alpha_{\lambda-1}, \dots, \alpha'_3 = \alpha_3, \alpha'_2 = \alpha_2 - 1$, and $\alpha_1 < \alpha - 2$.

On the other hand, when none of the α' 's is at least two units larger than the smallest α'_β which is less than α_β the number of subgroups of type $(\alpha'_1, \alpha'_2, \dots, \alpha'_\lambda)$ was proved above to be of the form $1 + p + kp^2$ provided there is at least one α' which exceeds α'_β . If there is no such α' there is only one subgroup of the given type. These results establish, in particular, the following:

THEOREM. *In any non-cyclic abelian group of order p^m the number of the subgroups whose order is a given proper divisor of the order of the group is always of the form $1 + p \bmod p^2$.*

UNIVERSITY OF ILLINOIS.

ON THE RECTIFIABILITY OF A TWISTED CUBIC.

BY PROFESSOR TSURUICHI HAYASHI.

UNDER the same title in the November number of this BULLETIN, 1918, volume 25, page 87, Dr. Mary F. Curtis showed that, given the twisted cubic

$$x_1 = at, \quad x_2 = bt^2, \quad x_3 = ct^3, \quad abc \neq 0,$$

the condition that it be a helix is precisely the condition that it be algebraically rectifiable. Since her proof is an application of the common differential geometry, the coordinates x_1, x_2, x_3 are rectangular. The parametrical equations of the most general twisted cubics in rectangular coordinates x, y, z , are

$$x = f_1(t)/F(t), \quad y = f_2(t)/F(t), \quad z = f_3(t)/F(t),$$

where F, f_1, f_2, f_3 are polynomials of degree 3 in the parameter t . By increasing t by a constant, and taking the axes along the tangent, the principal normal, and the binormal at the point on the curve where t is infinite, these equations are reduced to

$$(1) \quad x = \frac{a_0 t^2 + a_1 t + a_2}{t^3 + dt + e}, \quad y = \frac{b_1 t + b_2}{t^3 + dt + e}, \quad z = \frac{c_2}{t^3 + dt + e},$$

as Mr. W. H. Salmon has done in treating the twisted cubics of constant torsion.* But I will consider here the algebraic rectifiability of a less general type of twisted cubics whose equations are

$$x = a_1 t^3 + a_2 t^2 + a_3 t + a_4,$$

$$y = b_1 t^3 + b_2 t^2 + b_3 t + b_4,$$

$$z = c_1 t^3 + c_2 t^2 + c_3 t + c_4.$$

This type is even more general than that treated by Dr. Curtis, and contains the cubics which have the plane at infinity as their osculating plane.

By changing the value of t by a constant, transferring the origin, and taking the axes of coordinates along the tangent, the principal normal and the binormal at the point on the

* *Messenger of Mathematics*, vol. 45 (1915-1916), pp. 125-128.

curve where t is then equal to zero, the equations may be written

$$(2) \quad x = a_1 t^3 + a_3 t, \quad y = b_1 t^3 + b_2 t^2, \quad z = c_1 t^3.$$

From these equations we get

$$\Sigma \dot{x}^2 = (3a_1 t^2 + a_3)^2 + (3b_1 t^2 + 2b_2 t)^2 + (3c_1 t)^2,$$

$$\Sigma (\ddot{y}\ddot{z} - \dot{z}\dot{y})^2 = (6b_2 c_1 t^2)^2 + (6a_3 c_1 t)^2 + (-6a_1 b_2 t^2 + 6a_3 b_1 t + 2a_3 b_2)^2$$

and

$$\dot{x}\ddot{y}\ddot{z} = 12a_3 b_2 c_1.$$

The ratio of the radii of torsion and curvature

$$\frac{T}{R} = - \left(\frac{\Sigma (\ddot{y}\ddot{z} - \dot{z}\dot{y})^2}{\Sigma \dot{x}^2} \right)^{3/2} \cdot \frac{1}{|\dot{x}\ddot{y}\ddot{z}|},$$

can now be expressed in terms of t . Equating this ratio to a constant, we get only two conditions among the five coefficients $a_1, a_3; b_1, b_2; c_1$, which are necessary and sufficient in order that the twisted cubic be a helix:

$$(3) \quad b_1 = 0 \quad \text{and} \quad 9a_3^2 c_1^2 = 12a_1 a_3 b_2^2 + 4b_2^4.$$

Hence the theorem:—*In order that the twisted cubic of the type (2) be a helix it is necessary and sufficient that conditions (3) be satisfied.* The value of T/R for the helix is equal to $\pm 2b_2^2/3a_3 c_1$.

In this case,

$$\Sigma \dot{x}^2 = (9a_1^2 + 9c_1^2)t^4 + (6a_1 a_3 + 4b_2^2)t^2 + a_3^2,$$

so that the arc of the twisted cubic

$$s = \int_{t_0}^{t_1} (\Sigma \dot{x}^2)^{1/2} dt$$

is algebraic when and only when conditions (3) hold. Hence the theorem: *The twisted cubic of the type (2) is algebraically rectifiable when and only when it is a helix, a generalization of Dr. Curtis's theorem.*

Lastly I will add here a theorem on twisted cubics of constant torsion and of constant curvature. It can be shown that for the twisted cubic of the type (2) of constant torsion it is necessary and sufficient that

$$(4) \quad b_1 = 0, \quad a_1^2 + c_1^2 = 0, \quad 3a_3 c_1^2 = 2b_1 b_2^2.$$

But this is comprised in Mr. Salmon's theorem in his paper referred to. It can be also shown that for the twisted cubic of the type (2) of constant curvature, the same conditions (4) are necessary and sufficient. This was carried out by Mr. S. Narumi, one of our students. Hence the theorem: *The twisted cubics of the type (2) of constant torsion are of constant curvature, and conversely those of constant curvature are of constant torsion; they are all imaginary and satisfy the conditions (4).* A similar problem on the twisted cubics of the most general type (1), that is rather more interesting, as Prof. T. Kubota, one of our colleagues, says, remains unsolved.

February, 1919.

SOME GENERALIZATIONS OF THE SATELLITE THEORY.

BY PROFESSOR R. M. WINGER.

§1. *The Rational Cubic.*

In a forthcoming paper* in the *American Journal of Mathematics* the author considers the satellite line of the cubic and some related curves. The object of the present note is to point out how the principal results there obtained can be generalized.

For the rational cubic R^3 ,

$$x_1 = 3t^2, \quad x_2 = 3t, \quad x_3 = t^3 + 1,$$

the relation connecting a point τ and its tangential t is

$$t\tau^2 + 1 = 0.$$

And the condition that $3n$ points lie on a curve C_n is

$$s_{3n} = (-1)^n,$$

where s refers to the product of the t 's†. From these equations we have at once the following theorems:

1. *The tangentials of the points in which a C_n' cuts R^3 lie on a second C_n .*

* "On the satellite line of the cubic," read before the San Francisco section of the American Mathematical Society, April 6, 1918.

† Winger, "Involutions on the rational cubic," this BULLETIN, Oct., 1918.

2. *The contacts of tangents from the points in which a C_n meets R^3 lie on a C_{2n}' when n is even—and only then.*

Theorem 1 repeatedly applied yields a chain of curves of the same degree. We should naturally designate C_n' and C_n as primary and satellite respectively, but this terminology is hardly appropriate when $n > 2$. For then each curve is one of a pencil on the same points. We shall therefore speak of the points in question as a *primary set* S_{3n}' and a *satellite set* S_{3n} .

Any primary set has a unique satellite set* but a satellite set has a multitude of primaries. Let a conic C_2 cut the cubic in six points $t_1, t_2, \dots, t_6$. Then $s_6 = 1$. The tangentials of these points may be arranged in two sets $\tau_1, \tau_2, \dots, \tau_6$ and $-\tau_1, -\tau_2, \dots, -\tau_6$ such that the product of each set is unity. Call these points A and B and denote their products by A_6 and B_6 respectively. Selecting 6 t 's from points A and B in such a way that each subscript occurs but once, the set will lie on a conic provided it contains an even number of points B . There are thus 32 primary conics in all. Or, *the twelve contacts of tangents from the points of intersection of a conic and a rational cubic lie by sixes on thirty-two conics.*

In general, n even, a satellite set S_{3n} will have as many primary sets as there are combinations of $3n$ things 0, 2, 4, $\dots$, $3n$ at a time. The total number is therefore the sum of alternate binomial coefficients, beginning with the first, of degree $3n$,† i.e., 2^{3n-1} .

When n is odd, group the points τ so that $A_{3n} = 1$, $B_{3n} = -1$. Then points B but not A are on a C_n . We shall get a primary set as before, provided we use an odd number of points B . The total number of such sets therefore is the sum of alternate binomial coefficients of degree $3n$ beginning with the second, again 2^{3n-1} .

Hence in either case, *the $6n$ contacts of tangents from the intersections of C_n and R^3 can be arranged in primary sets S_{3n}' in 2^{3n-1} ways.*

§2. *The General Cubic.*

Theorem 1 of the previous section can be extended at once to the general cubic by the theory of residuation. Theorem 2 is replaced by

* If theorem 1 is applied to the points of intersection of R^3 and C_{3n}' of 2, the satellite set S_{3n} is an S_{3n} repeated.

† i.e., in the expansion of a binomial of degree $3n$.

3. The $12n$ contacts of tangents from the points in which a C_n meets the general cubic lie on a C'_{4n} .

For if the elliptic argument u is properly chosen, the intersections u_i of C_n satisfy the condition

$$u_1 + u_2 + \cdots + u_{3n} = \mu\omega + \mu'\omega'.$$

The contacts of tangents from u_i are

$$-\frac{u_i}{2}, \quad -\frac{u_i + \omega}{2}, \quad -\frac{u_i + \omega'}{2}, \quad -\frac{u_i + \omega + \omega'}{2},$$

the sum of which ($i = 1, \dots, 3n$) is $-2\sum u_i - 3n(\omega + \omega')$, hence the theorem is proved.

Again a primary set has a unique satellite set. To enumerate the primary sets of a satellite set, consider first a cubic C_3 meeting the base curve in nine points $u_1, u_2, \dots, u_9$. Denote the contacts of tangents from points u_i by A_i, B_i, C_i and D_i . A cubic C'_3 on eight of these points $\alpha_1, \dots, \alpha_8$,† where the subscripts are all different, will cut again in a ninth point, say u . The satellite set of C'_3 will be cut out by a cubic on the eight points $u_1, \dots, u_8$. But any cubic on these eight points passes through u_9 . Hence u_9 is the tangential point of u , or u is none other than a point α_9 . That is to say, any cubic on eight of the points α , so chosen that all subscripts except one are represented, will pass through one of the four points carrying the omitted subscript and will have points u_i for a satellite set. Now the points α_1 can be chosen in four ways, α_2 likewise and so on to α_8 , α_9 then being determined. Hence the points A, B, C, D can be chosen as primary sets in 4^3 ways. By precisely the same reasoning, the contacts of tangents from the $3n$ points in which a C_n cuts C_3 can be arranged in primary sets S_{3n}' in 4^{3n-1} ways.

§3. Extension to Higher Cases.

The satellite theory, as frequently happens, can be generalized in several directions. This is plain if it is recalled that there are three factors involved—the primary (a line), the base curve (a cubic), and the tangent lines at the points of intersection. Any one, any two or all three of these elements may be generalized. Of the more interesting cases we mention

* Pascal, Repertorium, 2d ed., II, p. 393.

† α_i refers to any one of the points with subscript i .

the following, the validity of which can be inferred from the fundamental theorem of residuation.*

4. *The n tangents at collinear points of a C_n meet again in $n(n-2)$ points which lie on a C_{n-2} .*

5. *The mn tangents to C_n at points of intersection of a C_m cut C_n again in points of a $C_{m(n-2)}$.*

6. *Given a base curve C_n , and a primary C_m , cutting C_n in points P_i . Then any set of mn C_p 's cut C_n in points of a C_{mnp} . If $C_p^{(i)}$ has k -point contact with C_n at P_i , the remaining $mn(np-k)$ points constitute a satellite set.*

Theorem 6 which is itself a special case of the fundamental theorem includes all the generalizations suggested. Theorems 4 and 5 repeatedly applied yield chain theorems connecting an unending series of curves.

In particular the dual of 5, beginning with $n=1$, establishes the chain, the first link of which was obtained by another method in the earlier paper.†

Many isolated classic theorems appear here as special cases. From 6 can be deduced readily the theorem that if $n-1$ intersections of a line with a C_n are points P_k with k -point tangents, $k>2$, the remaining intersection is a P_k . This includes as a special case the familiar theorem concerning flexes.

We shall conclude with a consideration of curves C_m each of which has with a base cubic a $(3m-1)$ -point contact P_{3m-1} . Thus conics with quintactic points at intersections of a line l' with a rational cubic cut again in three points of a line l . But from each of the latter points can be drawn five quintactic conics.‡ These 15 quintactic points lie by threes on 25 lines l' five of which pass through each point.

Similarly the 24 P_8 's of cubics passing simply through the intersections of l and R^3 determine a configuration of 24 points and 64 lines, 3 points on a line and 8 lines on a point.

Again if l is replaced by a conic C_2 there will be 30 quintactic conics from points of intersection of C_2 and R^3 . By reasoning analogous to that employed in §2 it is easily established that these 30 points lie by sixes on 5^5 conics C_2' .

Generally, from a point of R^3 can be drawn $(3m-1)$ C_m 's each having a contact P_{3m-1} elsewhere. The $3i$ common points

* Pascal, Repertorium, 2d ed., II, p. 321.

† Winger, "On the satellite line of the cubic," l.c.

‡ Winger, "Involutions on the rational cubic," l.c., p. 30.

of a C_i and R^3 determine thus $3i(3m-1)$ such points which can be arranged in primary sets S'_{3i} of the $3i$ points in $(3m-1)^{3i-1}$ ways.

Passing now to the general cubic, if a C_m have $(3m-1)$ -point contact at u' and cuts again at u the relation connecting the elliptic arguments is

$$u' = \frac{\mu\omega + \mu'\omega' - u}{3m-1}.$$

For given u there are $(3m-1)^2$ values of u' since $\mu, \mu' = 1, 2, \dots, 3m-1$. Thus from the common points of a conic and a C_3 can be drawn $6 \cdot 25$ quintactic conics whose 150 quintactic points lie by sixes on 5^{10} primary conics.

And generally from the $3i$ common points of a C_i and C_3 can be drawn $3i(3m-1)^2$ C_m 's each having a $(3m-1)$ -point contact. These $3i(3m-1)^2 P_{3m-1}$'s can be grouped in primary sets S_{3i}' of the $3i$ points in $3(3m-1)^{3i-2}$ ways.

UNIVERSITY OF WASHINGTON.

CAJORI'S HISTORY OF MATHEMATICS

A History of Mathematics. By FLORIAN CAJORI, Ph.D., Professor of the History of Mathematics in the University of California. New York, The Macmillan Company, 1919. vii + 514 pp. Price, \$4.00.

THE present edition of Cajori's well known History of Mathematics is so completely revised and so considerably enlarged that it might almost be regarded as a new book. While it contains only about 100 more pages than the earlier edition,* it has about twice as much reading matter, the pages being larger and more closely printed than those of the first edition. The general arrangement of the subjects treated remains unchanged, but three brief new sections relating largely to ancient mathematics have been added. These are headed; "The Maya," "The Chinese" and "The Japanese" respectively, and are based on special histories relating to these peoples, which have appeared since the publication in 1894 of the first edition of the present work.

* Reviewed in this BULLETIN, vol. 3 (1894), pp. 190 and 248, by D. E. Smith and G. B. Halsted.

Naturally the most important and extensive changes appear in the sections devoted to developments made during the nineteenth century. Somewhat less than one half of the space is devoted to advances made during the nineteenth and the early part of the twentieth century. To the younger mathematical students it is likely to be very interesting and useful to find here references to the work of so many contemporary mathematicians, and to find also brief statements of the relations between contemporary developments and those due to earlier investigators.

In view of the great variety of modern mathematical advances it cannot be expected that all of them could be treated in a one-volume history. The selection of the material to be included in such a volume was a very difficult undertaking and few would be likely to agree as regards details in making such a selection. In a broad way, however, probably most mathematicians will agree that Cajori succeeded very well in selecting his material and that he treated this material with an evident effort to exhibit relative values in a true light.

To secure reliable judgments as regards the bearing and value of much of the modern work, Cajori had to depend largely on the views of specialists. He quotes these views freely and sometimes extensively. The lack of uniformity in style and conservatism thus introduced is more than compensated by the expert knowledge relating to the wide range of fields of investigation which is thus made available. As a rule this expert knowledge has been wisely utilized and references are furnished. In some cases the latter are lacking but the total number of references is large and in this respect the present edition exhibits also a great improvement on the former one.

Two important questions about a history of mathematics are: Is the work written in an attractive style? and Is it confined to established facts? In the present case our answer to the former question is yes, and to the latter, no. Our negative answer to the second of these two questions does not imply a criticism of the book, since many readers will doubtless enjoy such speculations as those in regard to the reason why the ancient Babylonians divided the circle into 360 equal parts, page 6, as well as the references to reports which are as unlikely to be true as the one relating to the sacrifice of one hundred oxen by Pythagoras, page 18.

The value of a comparatively brief general history of mathematics cannot be measured entirely by its accuracy as regards details. The field to be covered is so extensive and the number of data is so large that it is almost impossible to avoid entirely the appearance of some misleading statements. What is most important is that the main points of view are sound and that the most fundamental factors in the development of our subject are properly emphasized.

The present work appears to be satisfactory along these important lines, as might have been expected from the fact that its author has had such a broad training in writing on the history of mathematics. Many historical questions present great difficulties and call for a wide knowledge of facts as well as a deep insight into the bearing of one result on another and the tendencies, local as well as national, to claim undue credit for the work of particular investigators.

It is interesting to note that in the present edition the word *khet* is used in place of *ruth* to denote an ancient Egyptian unit of unknown length employed in the work of Ahmes. The introduction of the term *ruth* for this unit seems to be due to a very crude error committed by J. Gow in his *History of Greek Mathematics*, 1884, page 127. In making use of A. Eisenlohr's translation of the work by Ahmes, Gow apparently failed to notice that the German word *Ruthen* means rods, and hence he translated this word by the word *ruths* instead of by the word *rods*.

The error thus committed found its way into other histories as well as into textbooks. It appears, for instance, on page 44 of the earlier edition of the present work as well as in the revised edition of Cajori's *History of Elementary Mathematics*, published in 1917. Fortunately, it is corrected in the present work and it is hoped that this correction may help to banish this error from the future textbooks on elementary geometry. In making the correction Cajori wisely transliterated the ancient Egyptian word instead of attempting to give an English equivalent.

Those who are especially interested in American mathematical work will be pleased to find so many references in the volume under review to the work done by Americans. By glancing over the Index it becomes at once apparent that America is on the mathematical map,—a fact which is not always revealed by the Index of a mathematical history. On

the other hand, one fails to find in this Index as large a number of references after the name of an American as appears after the names of various foreign mathematicians. This difference does not apply only to the deceased mathematicians, although it is more pronounced in these cases. In special instances such differences may be more or less due to accident, but when wide differences of this kind present themselves as regard large collections they should not be attributed to accident and their significance becomes a matter for serious consideration.

The semi-biographical feature of the earlier edition has been retained in the present one, as well as the use of heavy type in printing the names of some of the most eminent mathematicians. The following eight names of Americans appear in such type: M. Bôcher, J. W. Gibbs, G. M. Green, G. W. Hill, E. McClintock, A. Macfarlane, S. Newcomb, and B. Peirce. While no names of living American mathematicians are printed in this way, such a restriction has not been observed in regard to European mathematicians, as the following names appear in heavy type: P. Appell, P. Bachmann, E. I. Fredholm, G. Frege, F. Klein, H. Lebesgue, K. Pearson, E. Picard, H. A. Schwarz. Several of these names could have been replaced by those of more eminent living investigators.

One of the most useful things that the reviewer of a new edition of a popular textbook can do is to direct attention to possible improvements in such a way as to be helpful to those who will use the book in the class-room. It is with this object in view that we make the following suggestions: The note on page 98 relating to our lack of knowledge as regards the origin of our common numerals appears to the reviewer to be correct, but it does not seem to be in accord with a number of statements appearing in other parts of the book, e.g., pages 55, 88, 97, and 100. It seems as if this note had been added after it was too late to revise the other statements relating to this subject. If this is the case the note could have been made much more effective by stating this fact therein even if it might appear as acknowledging that the book was in need of a revision before it was published.

On page 27 it is stated that "Menaechmus invented the conic sections," and on page 39 we read that "Menaechmus, and all his successors down to Apollonius, considered only sections of *right* cones by a plane perpendicular to their sides." The latter of these two statements seems to be especially

misleading, since both Euclid and Archimedes knew how to generate an ellipse by cutting the cone by a plane *not* perpendicular to a generator. As regards the former statement, F. Dingeldey remarked that it is probably impossible to establish the correctness of the view that Menaechmus was the discoverer of conic sections and that it is very questionable whether he knew that the curves in question can be obtained by cutting a cone by means of a plane (*Repertorium der höheren Mathematik*, volume 2 (1910), page 197; *Encyclopédie des Sciences Mathématiques*, tome 3, volume 3, page 41).

That Cajori's language is not always clear and direct may be illustrated by the following quotation from page 47: "The fundamental theorem of plane trigonometry, that two sides of a triangle are to each other as the chords of double the arcs measuring the angles opposite the two sides, was not stated explicitly by Ptolemy, but was contained implicitly in other theorems." It is difficult to see why such cumbersome language is employed here to express the well-known law of sines in regard to the plane triangle. If the object was to illustrate the language which Ptolemy might have employed to express this law, reference should have been made to this fact.

Other instances of a lack of clarity or of obvious misprints may be found as follows:

Page.	Line.	Page.	Line.	Page.	Line.
49	31	299	18	372	39
239	27	302	16	373	34
250	21	327	1	377	20
"	22	330	10	401	32
265	31	331	8	418	22
267	33	351	35	438	15
287	3	360	33	494	42
296	6	361	26	497	37

To test the Index as regards completeness, the places where the term "group" would be likely to appear were looked up by the reviewer and it was found that the following page numbers should be added to those following this term in the Index: 318, 323, 390, 432. On the other hand, the term group does not appear on any of the pages 362-366 and hence the series of numbers 349-366 following this term in the Index should be replaced by 349-361. The name of H. Lebesgue does not appear in the Index, although it is printed in heavy type in the text, as was noted above.

It is singular that the Index does not contain the term "fundamental theorem of algebra." This term appears on page 363 of the text and reference is made in the Index to the substance of the theorem under the term "roots." To this reference there should be added the number 253 where the following sentence appears: "In the *Résolution des équations numériques* (1798) he (Lagrange) gave, among other things, a proof that every equation must have a root,—a theorem which before this usually had been considered self-evident."

This statement is clearly misleading, since various demonstrations of the theorem had been attempted long before 1798. In particular d'Alembert published such a supposed demonstration in 1746 and the theorem is still frequently called D'Alembert's theorem, especially in France. Among others who had attempted to give demonstrations before 1798 is L. Euler, who furnished two such demonstrations in 1749. In fact, Lagrange had also attempted to give such demonstrations more than twenty years before 1798. The great historical importance of the fundamental theorem of algebra makes it especially desirable that the references relating thereto should be explicit and accurate.

On page 234 we read "the calculus of variations to the invention of which Euler was led by the study of the researches of Johann and Jakob Bernoulli." On the other hand, the following sentence appears on page 251. "Lagrange did quite as much as Euler towards the creation of the Calculus of Variations." If Lagrange did quite as much as Euler towards the creation of this subject it is difficult to see why Euler should be regarded as its inventor.

The lack of uniformity in the two statements noted in the preceding paragraph would be less objectionable if Euler were commonly regarded by the best modern writers as the founder of the calculus of variations. On the other hand, the vagueness involved in the term inventor of a subject and the disagreement among historical writers as to who should be regarded as the founders of various subjects tend to mitigate somewhat such apparently contradictory statements as those noted above.

There is a very noticeable lack of uniformity as regards names. That both of the names Petrograd and St. Petersburg are used for the same city is perhaps due to the somewhat recent change of its name. This is less confusing

than the adoption of spellings according to different languages of the name of the same city and the use of different initials in referring to the same person at different places. For instance, those acquainted with the works of E. Lucas will be apt to be somewhat annoyed to find his name also in the form François Edouard Anatole Lucas. It is questionable whether anything is gained by giving G. Frege's name in the form Friedrich Ludwig Gottlob Frege.

The question of form as regards names in references and in historical works presents serious difficulties in view of the lack of uniformity along this line on the part of good writers. It is frequently very undesirable to give only the family name since the number of family names represented by different mathematicians is already large and is constantly growing larger. On the other hand the names in full are often so lengthy as to become burdensome. Perhaps the most satisfactory solution is to follow the example of the large mathematical encyclopedias. At any rate, it would be desirable to have uniformity in the same work unless an explanation is made as regards the equivalent different forms used in such a work.

Since B. Peirce occupies such a prominent place in the history of American mathematics it may be of interest to quote the following from page 338: "Profound are his researches on Linear Associative Algebra. The first of several papers thereon was read at the first meeting of the American Association for the Advancement of Science in 1864." The American Association for the Advancement of Science did not hold any meeting in 1864, its first meeting was held in 1848, and its *Proceedings* for this meeting do not mention any paper on linear associative algebra.

These suggested improvements may serve to direct attention to the fact that the careful student of the history of mathematics will be inclined to make a large number of marginal notes in Cajori's book and should be encouraged to do so. On the other hand, he will find here a very useful and readable history of our subject. The numerous quotations from reliable sources, the large number of references to the work of contemporary mathematicians and the fine spirit which pervades the whole work tend to increase the value of the book. It is the largest and most modern general history of mathematics in our language and we wish for it a success commensurate with the large amount of labor involved in its preparation.

UNIVERSITY OF ILLINOIS.

G. A. MILLER.

SHORTER NOTICES.

Mathematische Spiele. By W. AHRENS. Third edition. [Aus Natur und Geisteswelt, No. 170.] Teubner, Leipzig and Berlin, 1916. vi + 114 pp.

THE principal new matter introduced in preparing the third edition of this booklet consists of an additional section (Magische Quadrate auf Amuletten) in the chapter on magic squares and the enlargement of the chapter on mathematical fallacies, the latter having been increased from seventeen to twenty-eight pages. To make room for this without increasing the size of the book, the whole of Chapter IV of the second edition (on Wanderungsspiele) has been omitted and also the following parts of chapters: Chapter II, §4 (Das Puzzle mit Schranken); Chapter V, §1 in part (on the number of one's ancestors); Chapter VII, §4 (Eine andere Ankleidung des Spiels Nim). Otherwise only slight changes have been made. The modifications result in increasing the interest of the booklet.

R. D. CARMICHAEL.

Tratado Elemental de Goniometria. Por J. DE MENDIZABAL TAMBORREL. Second edition. Mexico, Departamento de Talleres Graficos de la Secretaria de Fomento, 1917. 209 pp.

THIS textbook contains an elementary treatment of the theory and applications of the trigonometric functions. It does not correspond closely with any of the usual textbooks on this subject as we know them in this country, the difference arising primarily from the fact that the author is willing to use freely the elementary methods of the differential calculus whenever the occasion suggests it. The first of the three parts of the book is devoted to a study of the trigonometric functions; the second and third parts, to applications involving plane triangles and spherical triangles respectively. The exposition is usually suited to the needs of a learner at the stage of his development generally found among our sophomores. The book is sometimes to be criticized for lack of rigor or completeness in the proofs. A glaring instance of this sort of defect is that afforded by the derivation of the usual infinite products for $\sin u$ and $\cos u$ on pages 86 and 87.

R. D. CARMICHAEL.

NOTES.

THE fourth summer meeting of the Mathematical Association of America, held at Ann Arbor in connection with the meetings of the American Mathematical Society and the American Astronomical Society, was attended by about ninety members. The meeting opened with a joint session of the three societies. The following day was devoted to discussion of the proposed mathematical dictionary and to a symposium on present day relations and tendencies between colleges and high schools as regards mathematics. The meeting closed with an exhibition of the mathematical and astronomical rarities in the library of the University of Michigan.

THE July number (volume 20, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "Certain types of involutorial space transformations," by F. R. SHARPE and VIRGIL SNYDER; "On a new treatment of theorems of finiteness," by O. E. GLENN; "On the theory of developments of an abstract class in relation to the calcul fonctionnel," by E. W. CHITTENDEN and A. D. PITCHER; "On the influence of keyways on the stress distribution in cylindrical shafts," by T. H. GRONWALL; "Some convergent developments associated with irregular boundary conditions," by J. W. HOPKINS; "Groups possessing a small number of sets of conjugate operators," by G. A. MILLER.

THE List of Members of the London mathematical society, dated November 1, 1919, includes 333 names, among them those of 16 honorary members. J. E. CAMPBELL is president and T. J. I'A. BROMWICH and G. H. HARDY are secretaries of the society.

THE following advanced courses in mathematics are offered at the Italian universities during the academic year 1919-1920:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Mechanics of continuous media with application to elastic bodies and to perfect and viscous fluids, three hours.—By Professor L. DONATI: Theory of heat (conduction, radiation,

thermodynamics, atomistics); principle of relativity and its recurrence in several questions of modern physics, three hours.—By Professor F. ENRIQUES: Abelian integrals, three hours.—By Professor S. PINCHERLE: Theory of sets of points; functions of a real variable and their integrals; theorems of existence for ordinary differential equations, three hours.

UNIVERSITY OF CATANIA.—By Professor M. CIPOLLA: Theory of elliptic functions with applications, four hours.—By Professor E. DANIELE: Mathematical theory of vibrations, four hours.—By Professor M. PICONE: Integral equations; potential; totally elliptic differential equations of mathematical physics, five hours.—By Professor G. SCORZA: Geometry on a curve from the algebraic-arithmetical standpoint of Dedekind and Weber, three hours.

UNIVERSITY OF GENOA.—By Professor G. LORIA: Geometry of hyperspaces, three hours.—By Professor C. SEVERINI: Calculus of variations, four hours.—By Professor O. TEDONE: Optical phenomena of higher order: absorption and dispersion of light, three hours.

UNIVERSITY OF MESSINA.—By Professor P. CALAPSO: Functions of a complex variable; elliptic functions, four hours.—by Professor G. GIAMBELLI: Differential theory of singularities of plane algebraic curves (following Enriques); projective geometry of hyperspaces; theory of moduli, three hours.—By Professor O. LAZZARINO: Electrostatics, four hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics in the middle ages (from 1200 to 1600 A.D.), three hours.—By Professor A. DEL RE: n -dimensional extensive analysis with applications to differential geometry and mechanics, four and one half hours.—By Professor R. MARCOLONGO: Relativity, three hours.—By Professor D. MONTESANO: Rational surfaces; birational correspondences between points of ordinary space, three hours.—By Professor E. PASCAL: Monogenic functions, three hours.—By Professor L. PINTO: Geometrical optics and the theory of optical instruments, three hours.

UNIVERSITY OF PADUA. By Professor U. AMALDI: Intro-

duction to the theory of integration of Sophus Lie, four hours.—By Professor F. D'ARCAIS: Harmonic functions; functions of a complex variable; Fourier's series, four hours.—By Professor P. GAZZANIGA: Theory of numbers, three hours.—By Professor G. RICCI: Absolute differential calculus; elasticity, four hours.—By Professor F. SEVERI: Algebraic geometry, especially of the rational surfaces, four hours.—By Professor A. TONOLO: Theorems of existence for ordinary and partial differential equations, four hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: Calculus of variations for the functions of one independent variable; integral equations, three hours.—By Professor M. DE FRANCHIS: Geometry on algebraic curves, three hours.—By Professor M. GEBBIA: Electromagnetism with particular regard to electric oscillations, four and one half hours.—By Professor A. SIGNORINI: Hydrodynamics, three hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: Geometry on an algebraic curve, three hours.—By Professor U. CISOTTI: Elasticity: distortions, finite strain, three hours.—By Professor F. GERBALDI: Functions of a complex variable and elliptic functions, three hours.—By Professor G. VIVANTI: Theory of algebraic equations, three hours.

UNIVERSITY OF PISA.—By Professor E. BERTINI: Cremona's transformations in plane and space, three hours.—By Professor L. BIANCHI: Differential geometry, three hours.—By Professor G. A. MAGGI: Physical optics, both from the elastic and the electromagnetic standpoint, four and one half hours.—By Professor ———: Mechanics (advanced), three hours.

UNIVERSITY OF ROME.—By Professor G. BISCONCINI: Geometrical applications of the calculus, three hours.—By Professor E. BOMPLANI: Geometry of applicability for surfaces and varieties, three hours.—By Professor F. CANTELLI: Calculus of probabilities with applications, three hours.—By Professor G. CASTELNUOVO: Non-euclidean geometry with physical interpretations, three hours.—By Professor U. CRUDELI: Introduction to advanced theories of electricity and magnetism, three hours.—By Professor T. LEVI-CIVITA: Curves defined by differential equations; periodic solutions, three hours.—

By Professor A. PERNA: Elementary theories of mathematical analysis, three hours.—By Professor L. SILBERSTEIN: Principle of relativity, three hours.—By Professor L. SILLA: Differential equations of dynamics, three hours.—By Professor V. VOLTERRA: General relativity, three hours; equations of mathematical physics, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Analytical dynamics, three hours.—By Professor C. SEGRE: Groups of finite order, three hours.—By Professor C. SOMIGLIANA: Theory of electricity and magnetism, three hours.—By Professor ———: Higher analysis, three hours.

PROFESSOR S. PINCHERLE, of the University of Bologna, has been elected corresponding member of the Royal institute of Venice.

PROFESSORS G. CASTELNUOVO and T. LEVI-CIVITA, of the University of Rome, have been elected corresponding members of the Royal academy of sciences of Bologna.

PROFESSOR E. LAURA, of the University of Pavia, has been promoted to a full professorship of rational mechanics.

PROFESSOR E. BORTOLOTTI, of the University of Modena, has been appointed professor of algebra and analytic geometry at the University of Bologna.

PROFESSOR W. H. YOUNG, of the University of Liverpool, has been appointed professor of mathematics at the University College of Wales.

DR. S. CHAPMAN, chief assistant at the Greenwich Observatory, has been appointed professor of mathematics at the University of Manchester.

IN October, Professor VITO VOLTERRA, of the University of Rome, lectured at the University of California on "Propagation of electricity in a magnetic field" and "Equations involving functional derivatives." Later he will deliver a course of three lectures at the Rice Institute on "Functions of composition."

THE Royal institute of Venice has awarded the Querini-Stampalia prize to Professor G. D. BIRKHOFF, of Harvard University, for his papers: "The restricted problem of three bodies," *Rendiconti del Circolo Matematico di Palermo*, volume 39 (1915), and "Dynamical systems with two degrees of freedom," *Transactions of the American Mathematical Society*, volume 18 (1917). The subject proposed was to realize some important advance in the theory of periodic solutions of differential equations.

DR. T. M. PUTNAM, professor of mathematics and dean of the undergraduate division of the University of California, has been appointed acting dean of the college of letters and science.

DR. E. V. HUNTINGTON, associate professor of mathematics in Harvard University, has been promoted to a full professorship in mechanics. His teaching activities will be divided as heretofore between the division of mathematics and the division of engineering.

PROFESSOR J. W. GLOVER, of the University of Michigan, served as acting president of the Teachers insurance and annuity association of America during September, in the absence of President Pritchett. Professor Glover is one of the trustees of the association.

DR. T. H. GRONWALL has been appointed mathematics and dynamics expert on the technical staff of the ordnance office, U. S. war department.

PROFESSORS P. BOUTROUX and J. H. M. WEDDERBURN both return from military service to Princeton University at the opening of the present academic year.

PROFESSOR H. C. WOLFF has been appointed head of the department of mathematics at the Drexel Institute.

AT the University of Illinois, assistant professor ARNOLD EMCH has been promoted to an associate professorship of mathematics. Dr. J. R. KLINE, of Yale University, and Dr. W. W. KÜSTERMANN, of the University of Michigan, have been appointed associates, and Mr. E. H. VANCE instructor in mathematics.

PROFESSOR IDA BARNEY, of Lake Erie College, Painesville, Ohio, has been appointed head of the department of mathematics at Meredith College, Raleigh, N. C.

PROFESSOR F. W. BEAL, of the University of Tennessee, has been appointed assistant professor of mathematics at the University of Pennsylvania.

DR. E. P. LANE, of the Rice Institute, has been appointed assistant professor of mathematics in the University of Wisconsin.

PROFESSOR W. V. N. GARRETSON, of the Pennsylvania Military College, has been appointed assistant professor of mathematics at Rutgers College.

DR. M. G. SMITH has been appointed professor of mathematics at Greenville College.

In the department of mathematics of Allegheny College, Mr. K. A. MILLER has been promoted to an assistant professorship and Mr. L. M. MONROE has been appointed instructor.

MR. H. R. PHALEN has been promoted to an assistant professorship of mathematics at the Armour Institute of Technology.

MR. A. C. MADDOX has been appointed assistant professor of mathematics at the Oklahoma Agricultural and Mechanical College.

DR. T. A. PIERCE has been appointed instructor in mathematics at the University of Nebraska.

DR. E. A. T. KIRCHER has resigned from the Sheffield Scientific School to enter business.

THE following instructors in mathematics have been appointed at Cornell University: Dr. F. W. REED, of the University of Illinois, Dr. G. M. ROBISON, and Mr. H. L. VANDIVER. Mr. A. D. CAMPBELL remains at Cornell, instead of going to Yale University as stated in the July BULLETIN.

DR. EUGENE TAYLOR has been appointed assistant professor of mathematics at the University of Wisconsin. Miss MARY A. COLPITTS has been appointed instructor in mathematics.

At Lehigh University, Mr. M. S. KNEBELMAN has been promoted to an assistant professorship of mathematics.

DR. J. V. MCKELVEY, of Cornell University, has been appointed assistant professor of mathematics in Iowa State College.

DR. FLORA E. LESTOURGEON, of Mt. Holyoke College, has been appointed assistant professor of mathematics in Carleton College, Northfield, Minn.

At the University of Pennsylvania, Dr. S. P. SHUGERT has been promoted to an assistant professorship of mathematics.

DR. TOBIAS DANTZIG, of Columbia University, has been appointed instructor in mathematics in Johns Hopkins University.

DR. L. M. KELLS, of the University of Illinois, has been appointed instructor in mathematics in the U. S. Naval Academy.

DR. S. D. ZELDIN has been appointed instructor in mathematics at the Massachusetts Institute of Technology.

MR. H. L. SMITH, of Cornell University, has been appointed instructor in mathematics at Trinity College, Hartford.

MR. I. A. ROMAN has been appointed instructor in mathematics at Northwestern University.

PROFESSOR EDUARDO TORROJA, of the University of Madrid, died September 14, 1918, in his seventy-second year.

PROFESSOR F. E. CHAPMAN, of the Gulf Coast Military Academy, died October 1, 1919.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- CASTRO ARAUJO (P.). *Analyse mathématique*. Rio Janeiro, 1917.
- CZUBER (E.). *Vorlesungen über Differential- und Integralrechnung*. 1ter Band. 4te Auflage. Leipzig, Teubner, 1918. 12 + 569 pp. Geb. M. 18.00
- DAHLGREN (T.). *Sur le théorème de condensation de Cauchy*. (Diss.) Lund, 1918. 4to. 69 pp.
- DOEHLEMANN (K.). *Projektive Geometrie in synthetischer Behandlung*. 4te verbesserte Auflage. Berlin, Göschen, 1918. M. 1.00
- DONADT (A.). See LÜBSEN (H. B.).
- HEIL (J.). *Das letzte Fermatsche Theorem und seine Lösung*. 2te verbesserte Auflage. Darmstadt, H. L. Schlapp, 1918.
- JUNKER (F.). *Höhere Analysis*. 1ter Teil: *Differentialrechnung*. 3te verbesserte Auflage. Berlin, 1918. M. 1.00
- KNOPP (K.). *Funktionentheorie*. 1ter Teil. 2te Auflage. (Sammlung Göschen, Nr. 668.) Leipzig, Göschen, 1918. 140 pp. M. 1.25
- LÖFFLER (E.). *Ziffern und Ziffernsysteme*. 1ter Teil: *Die Zahlzeichen der alten Kulturvölker*. 2te neu bearbeitete Auflage. Leipzig, 1918. M. 1.00
- LÜBSEN (H. B.). *Ausführliches Lehrbuch der analytischen und höheren Geometrie*. Zum Selbstunterricht. Neu bearbeitet von A. Donadt. 16te Auflage. Leipzig, 1919. M. 4.20
- MILNE (J.). *The analytical geometry of the straight line and the circle*. (Bell's Mathematical Series.) London, Bell, 1918. 12mo. 256 pp. 5 s.
- MINGOT SHELLEY (J.). *Aritmética razonada*. Granada, tip. Vazquez y Prieto, 1918. 8vo. 254 pp.
- REINHARDT (K.). *Ueber die Zerlegung der Ebene in Polygone*. (Diss.) Frankfurt a. M., 1918.
- REY PASTOR (J.). *Resumen de la teoria de las funciones analíticas y sus aplicaciones físicas*. Buenos Aires, 1918. 4to. 112 pp.

II. ELEMENTARY MATHEMATICS.

- BOREL (E.). *Die Elemente der Mathematik*. Deutsch von P. Stäckel. 1ter Band: *Arithmetik und Algebra*. 2te Auflage. Leipzig, Teubner, 1919. 404 pp. M. 11.00
- BÜRKLEN (O. T.). *Formelsammlung und Repetitorium der Mathematik*. 3te Auflage. Leipzig, 1918. M. 1.00
- CADOVIUS (I.), FISKE (A.), HERLEV (V.) og OLSEN (L. P.). *Regnebogen*. *Elevens Bog VII*, *Lærerens Bog VI*. København, Gyldendalske Boghandel, 1919.

- DIECK (W.). Stoffwahl und Lehrkunst im mathematischen Unterricht der Unter- und Mittelstufe höherer Lehranstalten. Leipzig, Teubner, 1918. 261 pp. M. 6.00
- FISKER (A.). See CADOVIVUS (I.).
- FRENKEL (E.). Tabellariisch overzicht der vlakke meetkunde met gebruiksaanwijzing. Utrecht, W. Leydenroth van Boekhoven, 1917. Fl. 0.35
- HART (W. W.). See WELLS (W.).
- HERLEV (V.). See CADOVIVUS (I.).
- LÖTZBEYER (P.). Vierstellige Tafeln zum logarithmischen und Zahlenrechnen für Schule und Leben. Leipzig, Teubner, 1918. M. 1.40
- LUNDGAARD (E.). Regnestokken. Kortfattet Vejledning til Indøvelse i dens Brug. København, Gjellerup, 1919. 28 pp.
- OLSEN (L. P.). See CADOVIVUS (I.).
- SCHÜLKE (A.). Aufgabensammlung aus der reinen und angewandten Mathematik. 1ter Teil, für die mittleren Klassen höherer Schulen. 3te Auflage. Leipzig, Teubner, 1917. 8vo. 8 + 204 pp. M. 2.60
- SMITH (D. E.). See WENTWORTH (G.).
- STÄCKEL (P.). See BOREL (E.).
- WELLS (W.) and HART (W. W.). New high school arithmetic, academic, industrial, commercial. Chicago, Heath, 1919. 8 + 358 pp.
- WENTWORTH (G.) and SMITH (D. E.). Higher arithmetic. Boston, Ginn, 1919. 6 + 250 pp. \$1.00

III. APPLIED MATHEMATICS.

- ARPESANI (C.). Elementi di tecnologia meccanica: lavorazione dei legnami. 2a edizione rinnovata ed ampliata. (Manuali Hoepli.) Milano, Hoepli, 1919. 24mo. 8 + 161 pp. L. 3.50
- CAMICHEL (C.), EYDOUX (D.) et GARIEL (M.). Etude théorique et expérimentale des coups de bélier. Paris, Dunod et Pinat, 1919. 4to. 398 pp.
- CASTLE (F.). A manual of machine design. London, Macmillan, 1919. 9 + 351 pp. 7s. 6d.
- DALE (J. H.). A course in machine drawing and sketching. London and Edinburgh, Chambers, 1919. 6 + 186 pp. 3s. 6d.
- EYDOUX (D.). See CAMICHEL (C.).
- FÖPPL (A.). Vorlesungen über technische Mechanik. 2ter Band: Graphische Statik. 4te Auflage. Leipzig, Teubner, 1918. 406 pp. M. 16.00
- FÖRSTER (G.). See REINHERTZ (C.).
- GARIEL (M.). See CAMICHEL (C.).
- GIORLI (E.). L'aritmetica e la geometria dell'operaio per le scuole professionali, d'arti e mestieri, tecniche, ferroviarie ed ad uso dei capi operai ed operai. 6a edizione ampliata. (Manuali Hoepli.) Milano, Hoepli, 1919. 24mo. 12 + 239 pp. L. 4.00
- HANSTEIN (R. VON). See POSKE (F.).

- HAUBER (W.). Festigkeitslehre. 5ter Neudruck. Berlin, 1918. M. 1.00
 ——. Statik. 2ter Teil: Angewandte Statik. 4ter Neudruck. Berlin, 1918. M. 1.00
- LÖTZBEYER (P.). Grundlagen der darstellenden Geometrie mit Einschluss der Perspektive. Dresden, Ehlermann, 1918. 132 pp. M. 0.45
- MACH (E.). Ueber Erscheinungen an fliegenden Projektilen. Vom räumlichen Sehen. Zwei Vorträge. Leipzig, Barth, 1917. 31 pp. M. 0.45
- MÜLLER (E.). Lehrbuch der darstellenden Geometrie für technische Hochschulen. 1ter Band. 2te Auflage. Leipzig, Teubner, 1918. 370 pp. Geb. M. 18.00
- MÜLLER (F. J.). Studien zur Geschichte der theoretischen Geodäsie. Als Vorarbeit zu einer Geschichte der Geodäsie. Augsburg, F. J. Müller, 1918. M. 6.50
- NERNST (W.) und SCHOENFLIES (A.). Einführung in die mathematische Behandlung der Naturwissenschaften. Kurzgefasstes Lehrbuch der Differential- und Integralrechnung mit besonderer Berücksichtigung der Chemie. 8te vermehrte und verbesserte Auflage. München, 1918. M. 17.00
- POSKE (F.) und HANSTEIN (R. von). Der naturwissenschaftliche Unterricht an den höheren Schulen. (Schriften des deutschen Ausschusses für den mathematischen und naturwissenschaftlichen Unterricht, 2te Folge, Heft 5.) Leipzig, Teubner, 1918. M. 1.40
- REINHERTZ (C.). Geodäsie. 2te Auflage. Neu bearbeitet von G. Förster. Berlin, 1918. M. 1.00
- SCHLICK (M.). Raum und Zeit in der gegenwärtigen Physik. Zur Einführung in das Verständnis der allgemeinen Relativitätstheorie. Berlin, Springer, 1917. 63 pp.
- SCHOENFLIES (A.). See NERNST (W.).
- SLATE (F.). The fundamental equations of dynamics and its main co-ordinate systems vectorially treated and illustrated from rigid dynamics. Berkeley, University of California Press, 1918. 7 + 233 pp.
- VATER (R.). Die neueren Wärmefunktionen. 1ter Band, 5te Auflage. 2ter Band, 4te Auflage. Leipzig, Teubner, 1918. 119 + 114 pp.
- VONDERHIM (J.). Schattenkonstruktionen. Neudruck. Berlin, 1918. M. 1.00
- WAGNER (P.). Die Stellung der Erdkunde im Rahmen der Allgemeinbildung. (Schriften des deutschen Ausschusses für den mathematischen und naturwissenschaftlichen Unterricht, 2te Folge, Heft 6.) Leipzig, Teubner, 1918. M. 1.00

Publications of the American Mathematical Society

Transactions of the American Mathematical Society.

Published quarterly. Subscription price for annual volume, \$5.00; to members of the Society, \$3.75.

Mathematical Papers of the Chicago Congress, 1893.

Price, \$3.00; to members, \$1.50.

Evanston Colloquium Lectures, 1893. By FELIX KLEIN.

Price, 75 cents.

Boston Colloquium Lectures, 1903. By H. S. WHITE, F.

S. WOODS and E. B. VAN VLECK. Price, \$2.00; to members, \$1.50.

Princeton Colloquium Lectures, 1909. By G. A. BLISS

and EDWARD KASNER. Price, \$1.50; to members, \$1.00.

Madison Colloquium Lectures, 1913. By L. E. DICKSON

and W. F. OSGOOD. Price, \$2.00; to members, \$1.50.

Cambridge Colloquium Lectures, 1916. Part I. By G

C. EVANS. Price, \$1.50; to members, \$1.00.

Address all orders to AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.

Whole No. 282

CONTENTS

	PAGE
The Twenty-Sixth Summer Meeting of the American Mathematical Society. By Professor E. J. MOULTON	49
Form of the Number of Subgroups of Prime Power Groups. By Professor G. A. MILLER	66
On the Rectifiability of a Twisted Cubic. By Professor TSURUICHI HAYASHI	73
Some Generalizations of the Satellite Theory By Professor R. M. WINGER	75
Cajori's History of Mathematics. By Professor G. A. MILLER	79
Shorter Notices	86
Notes	87
New Publications	94

Changes of address of members, exchanges, and subscribers should be communicated at once to the Secretary of the American Mathematical Society, 501 West 116th Street, New York.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to The American Mathematical Society, 41 North Queen St., Lancaster, Pa., or 501 West 116th Street, New York.

The initiation fees and annual dues of members of the American Mathematical Society are payable to the Treasurer of the American Mathematical Society, Professor J. H. TANNER, Cornell University, Ithaca, N. Y.

Articles for insertion in the BULLETIN should be addressed to the
BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.

The following dates have been fixed for the meetings of the Society :

Annual Meeting, New York: December 30-31, 1919

THE NEW ERA PRINT, LANCASTER, PA.

DEC 21 1919

LIBRARY

BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY

**A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE**

EDITED BY

F. N. COLE VIRGIL SNYDER J. W. YOUNG
ALEXANDER ZIWET D. E. SMITH
T. LEVI-CIVITA R. C. ARCHIBALD

VOLUME XXVI., NUMBER 3

DECEMBER, 1919

PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK
1919

Entered at the Post Office at Lancaster, Pa., as second class matter
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER

Whole No. 283

Digitized by Google **\$5.00 a year**

Taylor & Allen's **Junior High School Mathematics**

By E. H. TAYLOR AND FISKE ALLEN, of the
Eastern Illinois State Normal School

First Book. 210 pp. 12mo. 88 cents

Second Book. 251 pp. 12mo. 92 cents

ALL that is best and modern in the more recent development of the teaching of mathematics in the 7th and 8th grades is represented by these books. They assume that the pupil has mastered the work in arithmetic usually given in the first six grades. They continue the work in arithmetic by drill to obtain speed and accuracy. Percentage and its applications in modern business, as well as mensuration, are given special attention.

Algebra is approached through the formula, which is its most practical aspect for the beginning pupil. Practice in stating rules as formulas and formulas as rules is woven to such an extent into the fabric of the text that for the pupil it gradually becomes a natural expression of mathematical truths.

The examples are from the student's everyday life. He learns to figure the calorific value of a pound of sirloin steak or a pound of butter. Tables are furnished which teach the relative value of foods. Formulas are given whereby the pupil can figure out the horsepower of various makes of automobiles.

Groups of problems arranged in the form of projects are given the pupils to solve. A Boy's Dairy Project, A Boy's Project with Capons, and A Farm Project are examples of this.

HENRY HOLT AND COMPANY

**19 West 44th Street
NEW YORK**

**6 Park Street
BOSTON**

**2451 Prairie Ave.
CHICAGO**

NOTE ON CONVERGENCE TESTS FOR SERIES AND ON STIELTJES INTEGRATION BY PARTS.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society September 4, 1919.)

THE obvious identity

$$(1) \quad [c_k^{(1)}c_k^{(2)} \dots c_k^{(n)} - a_k^{(1)}a_k^{(2)} \dots a_k^{(n)}] = [a_k^{(2)}a_k^{(3)} \dots a_k^{(n)} (c_k^{(1)} - a_k^{(1)})] + \sum_{i=2}^{n-1} [c_k^{(1)} \dots c_k^{(i-1)} a_k^{(i+1)} \dots a_k^{(n)} (c_k^{(i)} - a_k^{(i)})] + [c_k^{(1)} \dots c_k^{(n-1)} (c_k^{(n)} - a_k^{(n)})]$$

may serve in several ways for the investigation of the convergence of series. In each it is to be used as a means of relating the $n+1$ sums of the first m terms of $n+1$ different series the general k th terms of which in order are the $n+1$ bracketed expressions in the foregoing identity. *It is obvious that the convergence of any n of these series implies the convergence of the remaining one.* Moreover, when they all converge, the identity leads to an obvious relation among the $n+1$ sums.

The effectiveness of the whole class of theorems arising thus lies in the fact that the series whose convergence is asserted may be, and in important cases is, conditionally convergent while the auxiliary series are absolutely convergent.

To arrive at a special instance of the theorems thus indicated, let us put $a_k^{(i)} = c_{k-1}^{(i)}$ with $c_0^{(i)} = 0$ and sum in (1) with respect to k from 1 to m . Then we have

$$(2) \quad c_m^{(1)}c_m^{(2)} \dots c_m^{(n)} = \sum_{k=1}^m c_{k-1}^{(2)}c_{k-1}^{(3)} \dots c_{k-1}^{(n)}(c_k^{(1)} - c_{k-1}^{(1)}) + \sum_{i=2}^{n-1} \sum_{k=1}^m c_k^{(1)} \dots c_k^{(i-1)} c_{k-1}^{(i+1)} \dots c_{k-1}^{(n)} \times (c_k^{(i)} - c_{k-1}^{(i)}) + \sum_{k=1}^m c_k^{(1)} \dots c_k^{(n-1)} (c_k^{(n)} - c_{k-1}^{(n)}).$$

From this relation it is easy to see that we have the following result:*

* The theorems of the paper are given in forms suitable for ready use where applicable. In view of the general remark in the first paragraph it is clear that stronger (though less simply stated) theorems may be obtained from identity (2). A similar remark may be made about other results in the paper.

THEOREM I. *The convergence of the series*

$$\sum_{k=1}^{\infty} c_{k-1}^{(2)} c_{k-1}^{(3)} \dots c_{k-1}^{(n)} (c_k^{(1)} - c_{k-1}^{(1)})$$

is implied by the convergence of the series

$$\sum_{k=1}^{\infty} |c_k^{(i)} - c_{k-1}^{(i)}|, \quad (i = 2, 3, \dots, n),$$

and the existence of the limits

$$\lim_{k=\infty} c_k^{(1)} c_k^{(2)} \dots c_k^{(n-1)},$$

$$\lim_{k=\infty} c_k^{(1)} \dots c_k^{(i-1)} c_{k-1}^{(i+1)} \dots c_{k-1}^{(n)}, \quad (i = 2, \dots, n-1).$$

In the special case when $n = 2$ let $c_k^{(1)}$ be the sum of the first $k-1$ terms of the converging infinite series $\alpha_1 + \alpha_2 + \dots$, so that $c_k^{(1)} - c_{k-1}^{(1)} = \alpha_{k-1}$. Then we have the following theorem: *If the series $\sum_k \alpha_k$ and $\sum_k |c_k^{(2)} - c_{k-1}^{(2)}|$ converge, so does also the series $\sum_k \alpha_k c_k^{(2)}$.* Owing to its wide and fundamental use in the theory of Dirichlet series, factorial series, and more general classes of series, this special case of our Theorem I must now be considered one of the classic tools of mathematical analysis.

The same identity (2) can be made to yield another general theorem of which the instance $n = 2$ is also classic. It may be stated as follows:

THEOREM II. *The convergence of the series*

$$\sum_{k=1}^{\infty} c_{k-1}^{(2)} c_{k-1}^{(3)} \dots c_{k-1}^{(n)} (c_k^{(1)} - c_{k-1}^{(1)})$$

is implied by the convergence of the series

$$\sum_{k=1}^{\infty} |c_k^{(i)} - c_{k-1}^{(i)}|, \quad (i = 2, 3, \dots, n),$$

the existence of the limit $\lim_{k=\infty} c_k^{(1)} c_k^{(2)} \dots c_k^{(n)}$, and the existence of a constant B (independent of k and i) such that for every k we have

$$|c_k^{(1)} c_k^{(2)} \dots c_k^{(n-1)}| < B, \quad |c_k^{(1)} \dots c_k^{(i-1)} c_{k-1}^{(i+1)} \dots c_{k-1}^{(n)}| < B \text{ for } i = 2, 3, \dots, n.$$

On taking $c_k^{(i)} = c_k$ for $i = 2, 3, \dots, n$, and writing ν for

$n - 1$, we have the following special case which deserves separate statement:

The convergence of the series

$$\sum_{k=1}^{\infty} (c_{k-1})^{\nu} (c_k^{(1)} - c_{k-1}^{(1)})$$

is implied by the convergence of the series $\sum_k |c_k - c_{k-1}|$ and either of the two following conditions:

(1) the existence of the limits

$$\lim_{k \rightarrow \infty} c_k^{(1)} (c_k)^{\nu-i} (c_{k-1})^{i-1}, \quad (i = 1, 2, \dots, \nu);$$

(2) the existence of the limit $\lim_{k \rightarrow \infty} c_k^{(1)} (c_{k-1})^{\nu}$ and the existence of a constant B (independent of k and i) such that for every k we have

$$|c_k^{(1)} (c_k)^{\nu-i} (c_{k-1})^{i-1}| < B, \quad (i = 1, 2, \dots, \nu).$$

If the quantities $c_k^{(i)}$ are not constants but are functions of a variable x on a linear interval or in a domain of the complex plane, then it is clear that the series whose convergence is asserted in these theorems will converge uniformly if the conditions in the hypotheses are satisfied uniformly. This means, in Theorem I, that the limits explicitly mentioned and those implied by the convergent series in the hypothesis shall exist uniformly. In the case of Theorem II it means that the limits stated or implied in the hypothesis shall exist uniformly and that the inequalities shall be valid for a B independent not only of k and i but also of x . The instance $n = 2$ of each of these theorems is now also to be considered classic.

If in (1) we set the a 's equal to unity and in the resulting identity sum as to k from 1 to m , we have

$$\begin{aligned} \sum_{k=1}^m (c_k^{(1)} c_k^{(2)} \dots c_k^{(n)} - 1) &= \sum_{k=1}^m (c_k^{(1)} - 1) \\ (3) \quad &+ \sum_{i=2}^n \sum_{k=1}^m c_k^{(1)} c_k^{(2)} \dots c_k^{(i-1)} (c_k^{(i)} - 1). \end{aligned}$$

From this we obtain the following theorem:

THEOREM III. *Suppose that the series*

$$\sum_{k=1}^{\infty} |c_k^{(i)} - 1|, \quad (i = 2, 3, \dots, n),$$

converge; then if either of the series

$$\sum_{k=1}^{\infty} (c_k^{(1)} c_k^{(2)} \cdots c_k^{(n)} - 1) \quad \text{and} \quad \sum_{k=1}^{\infty} (c_k^{(1)} - 1)$$

converges, the other does also.

A similar theorem for uniform convergence may be formulated in an obvious manner.

If we examine the first two of the foregoing theorems in relation to identity (2), we see that the hypothesis in each case is framed so as to secure simply the existence of the limit $\lim_{k \rightarrow \infty} c_k^{(1)} c_k^{(2)} \cdots c_k^{(n)}$ and the convergence of the series

$$(4) \quad \sum_{k=1}^{\infty} c_k^{(1)} \cdots c_k^{(n-1)} (c_k^{(n)} - c_{k-1}^{(n)}),$$

$$\sum_{k=1}^{\infty} c_k^{(1)} \cdots c_k^{(i-1)} c_{k-1}^{(i+1)} \cdots c_{k-1}^{(n)} (c_k^{(i)} - c_{k-1}^{(i)}),$$

($i = 2, 3, \dots, n-1$);

moreover, the existence of this limit and the convergence of these series are sufficient to secure the convergence of the series whose convergence is asserted. Now it will be observed that the series (4) are separately of the same form as that whose convergence is asserted in the first two theorems. Hence these theorems themselves may be used for testing the convergence of series (4); and similarly for the auxiliary series employed in applying these theorems to series (4). Thus we have a sequence of theorems for testing convergence. To state them in general in terms of a parameter would call for complicated formulas; these will not be given, since it is probably just as expeditious to use the given theorems repeatedly. For the case $n = 2$, however, the generalizations may be simply stated; these and others have been given in this BULLETIN, volume 20 (1914), pages 225-233 and in *Tôhoku Mathematical Journal*, volume 11 (1917), pages 191-199.

I was led to a consideration of identity (1) in connection with the problem of integration of a Stieltjes integral by parts. It is of interest to treat that problem in connection with the foregoing, particularly since it may be used to illustrate the fact that a generalization of Abel's transformation of series is afforded by identity (1).

Let π be a partition of the interval (ab) of norm δ due to the points $x_0 = a, x_1, x_2, \dots, x_{n-1}, x_n = b$ and let ξ_i be a point of (x_{i-1}, x_i) . We have the identity

$$\sum_{i=1}^n f(\xi_i) \{v_1(x_i)v_2(x_i) - v_1(x_{i-1})v_2(x_{i-1})\} \equiv \sum_{i=1}^n f(\xi_i)v_1(x_i)[v_2(x_i) - v_2(x_{i-1})] + \sum_{i=1}^n f(\xi_i)v_2(x_{i-1})[v_1(x_i) - v_1(x_{i-1})].$$

If we take $\xi_i = x_i$ and let δ approach zero, this relation yields the equation

$$(5) \quad \int_a^b f(x) d\{v_1(x)v_2(x)\} = \int_a^b f(x)v_1(x)dv_2(x) + \int_a^b f(x)v_2(x)dv_1(x),$$

provided that these Stieltjes integrals exist and that

$$(6) \quad \lim_{\delta \rightarrow 0} \sum_{i=1}^n v_2(x_{i-1})[f(x_i) - f(x_{i-1})][v_1(x_i) - v_1(x_{i-1})] = 0.$$

By taking $\xi_i = x_{i-1}$ and letting δ approach zero we have the same relation (5) provided that the integrals in it exist and that

$$(7) \quad \lim_{\delta \rightarrow 0} \sum_{i=1}^n v_1(x_i)[f(x_i) - f(x_{i-1})][v_2(x_i) - v_2(x_{i-1})] = 0.$$

Now if $f(x)$ is continuous on (ab) and $v_1(x)$ [$v_2(x)$] is of bounded variation on (ab) while $v_2(x)$ [$v_1(x)$] is bounded, it is obvious that (6) [(7)] is a valid relation. Thus we have sufficient conditions for the validity of (5) in case the integrals in (5) exist.

In particular, if $f(x)$ and either $v_1(x)$ or $v_2(x)$ are continuous on (ab) and both $v_1(x)$ and $v_2(x)$ are of bounded variation on (ab) , then the integrals in (5) certainly exist and that relation is a valid one.

Let $f(x)$ be identically equal to 1 and write $g(x)$ for $v_1(x)$ and $v(x)$ for $v_2(x)$. Then on interchanging the members in (5) we have

$$(8) \quad \int_a^b g(x)dv(x) + \int_a^b v(x)dg(x) = g(b)v(b) - g(a)v(a),$$

provided these integrals exist. This special case of (5) is one of the frequently used formulas in the theory of Stieltjes integration; it is a generalization of the theorem for integration by parts in the theory of Cauchy-Riemann integration.

The identity just preceding relation (5) is a generalization of that involved in the classic transformation of Abel; for, if we replace $v_1(x)$ by $g(x)$, $v_2(x)$ by $v(x)$, and $f(x)$ by 1, we have the Abel identity

$$\begin{aligned} \sum_{i=1}^n g(x_i)[v(x_i) - v(x_{i-1})] \\ \equiv - \sum_{i=1}^n v(x_{i-1})[g(x_i) - g(x_{i-1})] + g(b)v(b) - g(a)v(a). \end{aligned}$$

It is obvious that a repeated use of (5) reduces the integral of $f(x)$ as to a product $v_1(x)v_2(x) \cdots v_n(x)$ to a sum of n integrals of functions as to $v_1(x)$, $v_2(x)$, $\cdots$, $v_n(x)$, respectively, under appropriate conditions like (6) and (7) and the hypothesis of the existence of these n integrals. The question arises naturally as to whether some simple identity exists, analogous to that employed in deriving (5), which would yield the entire result at once. It was through this question that I was led to identity (1). Having it, it is natural to extend the classic theorems about convergence of series previously obtained through the particular case which yields Abel's transformation. The reader will have no difficulty in obtaining through identity (1) the transformation of a Stieltjes integral mentioned at the beginning of this paragraph.

UNIVERSITY OF ILLINOIS,
August, 1919.

NOTE ON A PHYSICAL INTERPRETATION OF STIELTJES INTEGRALS.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society September 4, 1919.)

STIELTJES was led to his definition of integral by what he called the problem of moments (see §24 of his memoir in *Annales de la Faculté des Sciences de Toulouse*, 1894). Consider on a straight line OX a distribution of (positive) mass, the mass m_i being concentrated at the distance ξ_i from the origin O . The sum $\sum m_i \xi_i^k$ be called the moment of order k of the mass with respect to the origin. He also considered the more general distribution of mass on OX which is such that

the mass on the segment $(0x)$ from 0 to x is $\varphi(x)$ where $\varphi(x)$ is a non-decreasing function which is finite for every finite value of x . Let us consider the moment about O , of order k , of the mass so distributed on the interval (ab) where a and b are both positive. Let π be a partition of (ab) of norm δ due to the points $x_0 = a, x_1, \dots, x_{n-1}, x_n = b$, and let ξ_i be a point of the interval (x_{i-1}, x_i) . Then for the moment in question we should obviously have the value

$$\lim_{\delta \rightarrow 0} \sum_{i=1}^n \xi_i^k \{ \varphi(x_i) - \varphi(x_{i-1}) \}.$$

That is, the value of the moment of order k is expressed by the Stieltjes integral

$$(1) \quad \int_a^b x^k d\varphi(x).$$

We thus have Stieltjes' physical interpretation of the integral (1) for each value of k , the function $\varphi(x)$ being monotonic.

The object of this note is to throw the physical interpretation of this integral into another form and to generalize the new form so as to afford a physical interpretation for the integral of $f(x)$ as to any function $v(x)$ of bounded variation.

Let us consider the curve $y = x^k$ and a distribution of mass on that curve such that the mass on the segment from $(0, 0)$ to (x, y) shall be $\varphi(x)$. Then the integral (1) represents the usual (first) moment of the mass on the segment of $y = x^k$ from $x = a$ to $x = b$ with respect to the axis of x , as one sees readily from the foregoing finite sum of which the integral (1) is the limit.

Now this interpretation of integral (1) admits of an obvious generalization. Let $f(x)$ be any single-valued function of x which is bounded on (ab) and let positive mass be distributed along the curve $y = f(x)$ so that the mass on the segment from $x = a$ to $x = x$ shall be $\varphi(x)$, where the functions $f(x)$ and $\varphi(x)$ have no common singularity and where the total mass involved is finite. Then the integral

$$\int_a^b f(x) d\varphi(x)$$

exists and gives the value of the moment about the x -axis of the mass on the curve $y = f(x)$ from $x = a$ to $x = b$.

Thus we have a physical interpretation for the integral of a bounded function with respect to any bounded monotonic non-decreasing function. If we avail ourselves of the notion of negative mass we may obtain similarly a physical interpretation for the integral of a bounded function $f(x)$ as to a function $v(x)$ of bounded variation.

Write $v(x)$ in the form

$$v(x) = v(a) + P(x) - N(x),$$

where $P(x)$ and $N(x)$ are respectively the positive and the negative variation of $v(x)$ on (ax) . Then along the curve $y = f(x)$ let us have a distribution of positive mass such that the positive mass on the segment from $x = a$ to $x = x$ is $P(x)$; along the same curve let us have a distribution of negative mass such that the negative mass on the segment from $x = a$ to $x = x$ is $N(x)$. We suppose that neither $P(x)$ nor $N(x)$ is discontinuous at a discontinuity of $f(x)$. Then the moment of the positive mass about the axis will be represented by the integral of $f(x)$ as to $P(x)$; that of the negative mass by the negative of the integral of $f(x)$ as to $N(x)$. Hence the algebraic sum of these moments, or the total moment about the x -axis, is represented by

$$(2) \quad \int_a^b f(x) dv(x).$$

Such a physical interpretation of the integral is useful in giving one a better intuitive sense of its character and hence in affording a means of classifying its properties in his thought.

If one is disturbed by the notion of negative mass which enters into the foregoing interpretation he may avoid its use by resorting to another range of physical phenomena. He may replace positive and negative mass by positive and negative magnetism. Then $P(x)$ [$N(x)$] would denote the total positive [negative] magnetism on the segment of $y = f(x)$ from $x = a$ to $x = x$ and the moment would be taken with respect to the x -axis under a magnetic field of force of unit intensity in a direction perpendicular to the xy -plane. The moment will again be represented by the integral (2).

It is obvious that this representation may also be modified into a geometrical one. Consider the cylindrical surface perpendicular to the xy -plane and intersecting it in the curve

$y = f(x)$. On the positive side of the xy -plane take the curve lying on this cylindrical surface and satisfying the condition $z = P(x)$; on the other side take the curve lying on the cylindrical surface and satisfying the condition $z = -N(x)$. We thus intercept between two curves a part of the cylindrical surface for which x is on (ab) . It is not difficult to form a fair intuitive notion of this portion of the surface since $P(x)$ and $N(x)$ are both monotonic non-decreasing. Then the integral (2) is the "area" of the part of this bounded cylindrical surface lying on the positive side of the xy -plane minus the "area" of that part lying on the other side.

UNIVERSITY OF ILLINOIS,
August, 1919.

A DERIVATION OF THE EQUATION OF THE NORMAL PROBABILITY CURVE.

BY PROFESSOR W. D. CAIRNS.

(Read before the American Mathematical Society September 5, 1918.)

THE symmetrical distribution of magnitudes about their mean is commonly represented by a "polygon" whose equally spaced ordinates are proportional to the terms of the expansion of $(1 + 1)^n$. The statement is frequently made in textbooks without any proof that as n is increased indefinitely, the equal spaces and the vertical scale being properly controlled, the polygon approaches as its limiting form the normal curve $y = ke^{-h^2x^2}$. The method here given for the proof of this theorem consists essentially in controlling what may be called the points of inflexion of the polygon so that these points approach predetermined positions on each side of the mean. Since an extended and rigorous proof of the probability theorem has been published,* it will suffice to indicate here the general plan of the proof.

* E. L. Dodd, *American Mathematical Monthly*, vol. 20 (1913), p. 128.

Since this paper was written, a proof by A. A. Bennett has appeared in this BULLETIN, volume 24, No. 10, page 477. In that proof the area (rather than the standard deviation, as used in the present paper) and the middle ordinate control the curve and Wallis's product formula for $\pi/2$ is used. Since the area and the standard deviation are alike fundamental in the applications to probability, statistics, theory of errors, etc., it would seem that each of these gives a natural method of approach.

By the elementary binomial formula the middle (greatest) term of $(1 + 1)^{2n}$ is $(2n)!/(n!)^2$; and by the shortened Stirling's formula

$$n! \sim \sqrt{2\pi} e^{-n} n^{n+\frac{1}{2}}$$

this term approximates for large values of n to $2^{2n}/\sqrt{\pi n}$.

If then we assign the terms of $\frac{\sqrt{\pi n}}{2^{2n}} (1 + 1)^{2n}$ to points on the x axis to the left and right of the origin at distances Δx apart, the middle term, assigned to the origin, approximates for large values of n to unity (and thus, by the introduction of a constant factor, to any preassigned value).

The k th term to the right (or left) is

$$\frac{\sqrt{\pi n}}{2^{2n}} \frac{(2n)!}{(n+k)!(n-k)!}$$

and this by Stirling's formula approximates to

$$\frac{\sqrt{n}}{2^{2n}} \frac{(2n)^{2n+\frac{1}{2}}}{\sqrt{2}(n+k)^{n+k+\frac{1}{2}}(n-k)^{n-k+\frac{1}{2}}},$$

i.e., to

$$\left(\frac{n-k}{n+k}\right)^k / \left(1 - \frac{k^2}{n^2}\right)^{n+\frac{1}{2}},$$

the form here used for our ordinate.

Aside from the factor $\sqrt{\pi n}(2n)!/2^{2n}$, the difference between the k th and the $(k+1)$ th term is

$$\frac{2k+1}{(n+k+1)!(n-k)!}.$$

Hence the k_σ th term for which the ordinates of the polygon decrease most rapidly is characterized by the condition

$$\begin{aligned} \frac{2k_\sigma - 1}{(n+k_\sigma)!(n-k_\sigma+1)!} &\leq \frac{2k_\sigma + 1}{(n+k_\sigma+1)!(n-k_\sigma)!} \\ &> \frac{2k_\sigma + 3}{(n+k_\sigma+2)!(n-k_\sigma-1)!}, \end{aligned}$$

or by an easy reduction

$$k_\sigma \leq \sqrt{(n+1)/2} < k_\sigma + 1.$$

If therefore as n increases we choose Δx so that $\sqrt{n/2} \cdot \Delta x$ equals σ , a predetermined positive constant, the last condition may be written

$$k\sigma\Delta x \leq \sigma\sqrt{(n+1)/n} < k\sigma\Delta x + \Delta x;$$

hence the k th term will with increasing n approach the position whose abscissa is σ . If further we set $k\Delta x = x$, i.e.,

$$k = \frac{x}{\sigma} \sqrt{n/2},$$

the k th ordinate, assigned to the position whose abscissa is x , will be

$$\frac{1}{\left(1 + \frac{x^2}{2\sigma^2 n}\right)^{n+1}} \left(\frac{n - \frac{x}{\sigma} \sqrt{\frac{n}{2}}}{n + \frac{x}{\sigma} \sqrt{\frac{n}{2}}} \right)^{\frac{x}{\sigma} \sqrt{\frac{n}{2}}}.$$

Finally, by evaluating this expression for $n = \infty$, it is found to approach uniformly the limit $e^{-x^2/2\sigma^2}$, whence the equation of the limiting form of curve is $y = e^{-x^2/2\sigma^2}$, the customary form of the normal curve equation save for an arbitrary constant factor.

From the foregoing it also follows that if instead of each ordinate there be used a rectangle of the same height and of base Δx , the sum of the areas of the rectangles approximates asymptotically to $\sqrt{\pi n} \Delta x$, i.e., to $\sigma \sqrt{2\pi}$; in other words, (1) the area under the curve is finite and (2) we have evaluated the integral

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx = \sigma \sqrt{2\pi}.$$

The writer has derived a series of formulas, a representative of which is the following:

$$\sum_{k=-n}^{k=n} \frac{(2n)!}{(n+k)!(n-k)!} k^2 = \frac{n}{2} 2^{2n};$$

i.e., if the binomial coefficients (or terms) of $(1+1)^{2n}$ be multiplied respectively by the squares of the number of the terms counting from the midterm, the sum of the products is as stated. (The theorem may also be stated in terms of

mean square deviation.) The use of this formula in the foregoing method gives the result

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} x^2 dx = \lim_{n \rightarrow \infty} \sum_{k=-n}^{k=n} \frac{(2n)!}{(n+k)!(n-k)!} (k\Delta x)^2 \Delta x = \sigma^3 \sqrt{2\pi}.$$

Similar evaluations are obtained for

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} x^4 dx, \quad \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} x^6 dx, \text{ etc.}$$

OBERLIN, OHIO.

BÔCHER'S BOUNDARY PROBLEMS FOR DIFFERENTIAL EQUATIONS.

Leçons sur les Méthodes de Sturm dans la Théorie des Équations Différentielles Linéaires et leurs Développements Modernes, professées à la Sorbonne en 1913-1914. Par MAXIME BÔCHER. Recueillies et rédigées par GASTON JULIA. Paris, Gauthier-Villars, 1917. 6 + 118 pp.

It can be said without fear of contradiction that what may be characterized as the *linear problem* is one of the most central in all mathematics. In algebra this problem concerns itself not only with linear forms and linear equations but also with many phases of the discussion of bilinear and quadratic forms. The results arrived at from an algebraic treatment find immediate application in geometry and mechanics. In the field of analysis the linear differential equation in one or more independent variables has always occupied a position of prime importance and in recent years the study of linear integral equations has not only forged a new and powerful tool but has also exerted a profound influence on the general trend of mathematical thought. The recent development of the theory of linear algebraic equations in an infinite number of unknowns by bridging the gap between the old algebraic field of linear equations and bilinear forms on the one hand and the analytic field of differential equations, integral equations, and bilinear forms in an infinite number of variables on the other, has given a remarkable unity to the various aspects of the general problem. In searching for the theory

underlying the various forms of the linear problem, Moore has been led to the development of his General Analysis.

Perhaps the most interesting questions of relative maximum and minimum and of calculus of variations are those intimately bound up with the solutions of problems arising in these other domains. The linear differential equation as well as the linear algebraic equation owes much of its importance to its connection with applied mathematics. In fact the original discussion by Sturm was suggested by problems arising in mathematical physics, and Bôcher himself, while a student at Göttingen in the late eighties and early nineties, was led into this field by problems of a like origin which were interesting Klein and his school.

Among the methods employed during recent years in the study of boundary problems, those of asymptotic expression and successive approximations pertain more or less strictly to the field of differential equations itself. It is, however, not surprising that other disciplines have been called in to aid in the development of the theory. Among the newer tools to be used in the attack might be included integral equations, calculus of variations, passage to the limit from a finite number of approximating linear equations in a finite number of variables, and linear equations in an infinity of variables. It was, however, one of the theses stoutly defended by Bôcher that the newer methods are more artificial and less elegant than those of Sturm and that they add little to the theory.* While admitting to some extent the force of this argument, it should be pointed out that the analogies suggested by the fresh points of view have been fruitful in indicating new problems and new avenues of advance even by the old methods. And it is especially worthy of note that while the Sturmiian methods proved incapable of solving boundary problems in two or more dimensions, the others lend themselves readily to such generalization.

In his article in the *Encyklopädie der Mathematischen Wissenschaften*, Professor Bôcher summarizes the main results attained in the discussion of boundary problems in the theory of ordinary differential equations before the present century. The paper which he read at the International Congress in 1912 outlined some of the lines of advance during the intervening years. As Harvard exchange professor at

* Cf. his lecture before the International Congress at Cambridge.

Paris, in the winter term of 1913-14, it was especially fitting that he lay stress on the notions and methods of Sturm, who in the same city more than three quarters of a century earlier had in a brilliant memoir laid the foundation of this fundamental theory. Soon after his student days Bôcher had conceived the idea of writing a treatise on boundary problems as an elaboration of his encyclopedia article and had entered into a contract with Teubner for its publication. A considerable portion of this manuscript in German was left among his papers, but the contract had been cancelled a few years before his untimely death. The present volume embodies much of the material he had prepared for the other project.

The contributions of Americans to this field added to the felicity of the choice of the topic for the Paris lectures. In addition to the many contributions of the lecturer and his students, several other Americans, who a decade or more ago came under the influence of Hilbert, have done investigation in this field. Besides the notable advances made in the several papers by Birkhoff, other names such as Mason, Westfall, Kellogg, and Hurwitz might be mentioned in this connection. It is worthy of note also that a considerable proportion of the articles* dealing with this topic and published

* Among the papers touching on this field published since Bôcher's survey in 1912 and not referred to in his *Leçons* are the following: J. Tamarin, "Sur quelques points de la théorie des équations différentielles linéaires ordinaires et sur la généralization de la séries de Fourier," *Rend. Cir. Mat. di Pal.*, vol. 34 (1912), pp. 345-382. Also see vol. 37 (1914), pp. 376-378 of same journal. G. D. Birkhoff, "Note on the expansion problems of ordinary linear differential equations," *Rend. Cir. Mat. di Pal.*, vol. 36 (1913), pp. 115-126. G. Hamel, "Ueber die lineare Differentialgleichung zweiter Ordnung mit periodischen Koeffizienten," *Math. Ann.*, vol. 73 (1913), pp. 371-412; also "Ueber das infinitäre Verhalten der Integrale einer linearen Differentialgleichung zweiter Ordnung, wenn die charakteristische Gleichung zwei gleiche Wurzeln hat," *Mathematische Zeitschrift*, vol. 1 (1918), pp. 220-228. J. Yoshikawa, "Miszellen aus dem Gebiete der Oszillationsaufgaben," *Memoirs of the College of Science and Engineering, Kyoto Imperial University*, vol. 5 (1913), pp. 97-115. O. Haupt, "Ueber eine Methode zum Beweise von Oszillationstheoremen," *Math. Ann.*, vol. 76 (1914), pp. 67-104. L. Lichtenstein, "Zur Analysis der unendlichvielen Variabeln. Entwicklungssätze der Theorie gewöhnlicher linearer Differentialgleichungen zweiter Ordnung," *Rend. Cir. Mat. di Pal.*, vol. 38 (1914), pp. 113-166. D. Jackson, "Algebraic properties of self-adjoint systems," *Trans. Amer. Math. Soc.*, vol. 17 (1916), pp. 418-424. T. Fort, "Linear Difference and Differential Equations," *Amer. Jour. of Math.*, vol. 39 (1917), pp. 1-26; also "Some theorems of comparison and oscillation," *Bull. Amer. Math. Soc.*, vol. 24 (1918), pp. 330-334. R. D. Carmichael, "Comparison theorems for homogeneous linear differential equations of general order," *Annals of Math.*, vol. 19 (1918), pp. 159-171. H. J. Ettlinger, "Existence theorems for the general real self-adjoint

since these lectures were prepared, are from the pens of Americans.

The relation of this volume to the rest of Bôcher's work has already been discussed in this BULLETIN* by Birkhoff in his general survey and estimate of the author's contributions to mathematics. The particular task which he has set himself in the book under review is that of expounding the theory of the linear differential equation, keeping in mind not only the spirit of Sturm's work but, so far as possible, the letter also. Choosing to neglect some of the more direct and powerful but less elegant methods, he has given an admirable exposition of the analogy of the problems of linear differential equations to those of the purely algebraic linear equations.

Ever since Sturm was inspired by these analogies to undertake his investigations, an advance in the linear problem of algebra or of analysis has generally suggested a corresponding one in the other field. Bôcher's contributions to the literature of both the algebraic and analytic problems made it especially fitting that he undertake this presentation. It should, however, be pointed out that it is purely the parallelism between the two theories which he discusses. No use is made of either theory in the actual development of the other. There is, for example, no hint of the possibility of obtaining results by passing from the algebraic to the transcendental on letting the number of difference equations approximating to the differential equation increase without limit.

Material from the published memoirs of Bôcher plays no inconsiderable part in the subject matter of this volume. And as Birkhoff has already pointed out, there are two important

linear system of the second order," *Trans. Amer. Math. Soc.*, vol. 19 (1918), pp. 79-96. W. B. Fite, "Concerning the zeros of the solutions of certain differential equations," *Trans. Amer. Math. Soc.*, vol. 19 (1918), pp. 341-352; also "The relation between the zeros of a solution of a linear homogeneous differential equation and those of its derivative," *Annals of Math.*, vol. 18 (1917), pp. 214-220. O. D. Kellogg, "Interpolation properties of orthogonal sets of solutions of differential equations," *Amer. Jour. of Math.*, vol. 40 (1918), pp. 225-234. R. G. D. Richardson, "Contributions to the study of oscillation properties of the solutions of linear differential equations of the second order," *Amer. Jour. of Math.*, vol. 40 (1918), pp. 283-316. C. E. Wilder, "Problems in the theory of ordinary linear differential equations with auxiliary conditions at more than two points," *Trans. Amer. Math. Soc.*, vol. 19 (1918), pp. 157-166; also see vol. 18 (1917), pp. 415-422 of same journal.

See also abstracts of two papers read before the meeting of the Amer. Math. Society in September 1919 by C. N. Reynolds, *Bull. Amer. Math. Soc.*, vol. 26 (1919), no. 2.

* February, 1919.

theorems which appear here for the first time. Below we shall discuss these theorems, one of which is concerned with an extension of the method of successive approximation and the other with the functional dependence of the solution on the coefficients of the equation and of the boundary conditions. A careful reading of these lectures gives the distinct impression also that a further filling in of gaps and a general rounding out of the theory is one of their admirable features.

The reader cannot fail to obtain from this volume an excellent insight into one of the fundamental bases of mathematical theory. It has the virtues and the faults of the lecture form, being interesting and suggestive but neither encyclopedic nor adapted for reference. It suffers from the fact that it was compiled by another; one notes the simplicity and elegance of ideas which were characteristic of Bôcher, but misses at times his lucidity of exposition. Sometimes the exact conditions under which a theorem is being developed are not easy to locate (pages 40, 71, 110) and it is occasionally difficult to find the formulation of results (pages 41, 42, 62, 81). The arrangement of material is not always the happiest and the passage from one topic to another is sometimes more abrupt than the division into sections would indicate. It may be that the war is in large measure responsible for a standard of proof-reading and typography somewhat lower than the very excellent one hitherto attained in the Borel series.

Before attempting to outline the argument of the text it may be well to consider briefly some of the properties of solutions of the special differential equation.

$$(1) \quad u'' + \lambda u \equiv d^2u/dx^2 + \lambda u = 0$$

and its approximating difference equations. For this special problem we shall not only, as Bôcher would do, set up the parallelism of the two theories but we shall go beyond him in exhibiting the analogy of the actual mechanism and in indicating how by passing to the limit one theory goes into the other. The results are fairly typical of the general homogeneous equation of the second order and will serve as an illuminating introduction to the general theory.

There are two linearly independent solutions of (1), viz.,

$\sin \sqrt{\lambda}x$ and $\cos \sqrt{\lambda}x$, depending on the parameter λ and giving as the most general solution $u = c_1 \sin \sqrt{\lambda}x + c_2 \cos \sqrt{\lambda}x$. The zeros of any two linearly independent solutions separate each other. If now one boundary condition is imposed, there can be only one linearly independent solution; for example, when $u(0) = 0$, the solution becomes $u = c_1 \sin \sqrt{\lambda}x$, having zeros at $\pm n\pi/\sqrt{\lambda}$. With increase of λ these zeros, or those of any solution, move closer together. It is readily seen that a second boundary condition can be imposed only in exceptional cases; the condition $u(1) = 0$ can, for example, be satisfied only if $\lambda = \pi^2, (2\pi)^2, \dots$. The oscillation theorem for equation (1) under the boundary conditions

$$(2) \quad u(0) = u(1) = 0$$

may be stated as follows: There exists one and only one characteristic number λ for which there is a non-identically vanishing characteristic solution of (1), (2) oscillating n times, that is, having n zeros in the interval. The infinity of positive characteristic numbers for $n = 1, 2, 3, \dots$ diverges to infinity.

By dividing the interval $(0, 1)$ into n equal parts and denoting by u_i the value at $x = i/n$, we can set up the algebraic equations approximating to (1) or (1) and (2). In the latter case they are

$$(3) \quad n^2(u_{i-1} - 2u_i + u_{i+1}) = -\lambda u_i, \quad u_0 = 0, u_n = 0 \\ (i = 1, \dots, n-1).$$

Essentially we have here $n-1$ homogeneous equations in $n-1$ unknowns which will have a solution only when λ is one of the $n-1$ roots of a determinant formed from the coefficients of the u 's. The λ -matrix of the coefficients which is the type met with in studying pairs of bilinear or quadratic forms is here symmetric. We can call the roots characteristic numbers and state for the corresponding solutions a theorem of oscillation analogous to that for the differential equation.

If, in the approximation equations (3), n is given the value 5 and the u 's on the left considered to be a different set of variables from the u 's on the right, and the former solved in terms of the latter, these equations take the form

$$\begin{aligned}
 u_1 &= \frac{\lambda}{25} (u_0 + \frac{4}{5}u_1 + \frac{3}{5}u_2 + \frac{2}{5}u_3 + \frac{1}{5}u_4 + 0u_5), \\
 u_2 &= \frac{\lambda}{25} (0u_0 + \frac{3}{5}u_1 + \frac{9}{5}u_2 + \frac{4}{5}u_3 + \frac{2}{5}u_4 + 0u_5), \\
 (4) \quad u_3 &= \frac{\lambda}{25} (0u_0 + \frac{2}{5}u_1 + \frac{4}{5}u_2 + \frac{9}{5}u_3 + \frac{3}{5}u_4 + 0u_5), \\
 u_4 &= \frac{\lambda}{25} (0u_0 + \frac{1}{5}u_1 + \frac{2}{5}u_2 + \frac{3}{5}u_3 + \frac{4}{5}u_4 + u_5), \\
 u_0 &= 0, \quad u_5 = 0.
 \end{aligned}$$

On denoting the range i/n ($i = 0, 1, 2, 3, 4, 5$) by x_i and also by ξ_i , the matrix of coefficients on the right may be written

$$\begin{aligned}
 (5) \quad K(x_i, \xi_j) &= (1 - \xi_j)x_i, \text{ for } x_i \leq \xi_j; \\
 K(x_i, \xi_j) &= (1 - x_i) \xi_j, \text{ for } x_i \geq \xi_j,
 \end{aligned}$$

and the solution (4) has the form

$$(4') \quad u_j = \frac{\lambda}{5} \sum_{i=1}^5 K(x_i, \xi_j) u_i \quad (j = 0, \dots, 5).$$

It will be noted that $K(x_i, \xi_j)$, which may be called the Green's function, is a symmetric matrix [$K(x_i, \xi_j) = K(\xi_j, x_i)$], and further that the first difference quotient $5[K(x_{i+1}, \xi_j) - K(x_i, \xi_j)]$ for each j is one constant for $i \leq j$ and another constant for $i > j$ and that the difference of these constants is in all cases unity. In other words, for $x_i = \xi_j$ the first difference of the matrix has a discontinuity equal to unity.

We may also set up a Green's function $K(x, \xi)$ for the differential equation (1) having the following properties: $K(x, \xi)$

- (a) is symmetric in the variables (x, ξ) ,
- (b) is continuous together with its first derivative, except that the latter has a break of 1 for $x = \xi$, and
- (c) satisfies the relation

$$(6) \quad u(x) = \lambda \int_0^1 K(x, \xi) u(\xi) d\xi.$$

This integral equation (6) is equivalent to the differential system (1), (2) and bears to it the same relation as equations

(4) and (4') do to equations (3). The Green's function

$$(7) \quad \begin{aligned} K(x, \xi) &= (1 - \xi)x \text{ for } x \leq \xi, \\ K(x, \xi) &= (1 - x)\xi \text{ for } x \geq \xi \end{aligned}$$

has essentially the same form (5) as that for the approximating algebraic equations (3).

The λ -matrix of any set of approximating equations such as (3) is equivalent to that of the corresponding set such as (4), both sets being taken as homogeneous. For any value of n , each of the characteristic numbers of (3) is equal to the corresponding one of (4). When n increases, each of the former approaches one of the infinite set of characteristic numbers $(n\pi)^2$ of (1), while each of the latter approaches one of the same infinite set of characteristic numbers of (6). The λ -determinant of (4) has for limit the symmetric Fredholm λ -determinant of (6). Thus by a passage to the limit $n = \infty$ the algebraic problem goes over completely into the transcendental.

After this consideration of a special problem, we are in a position to sketch the methods and results of the Leçons. Chapter II concerns itself with the parallelism between the linear differential and the linear algebraic systems. To a square array $|a_{ij}|$ there corresponds a set of linear homogeneous equations

$$(8) \quad a_{11}\xi_1 + a_{12}\xi_2 + \cdots + a_{1n}\xi_n = 0 \quad (i = 1, 2, \dots, n)$$

and also another closely related adjoint set

$$(9) \quad a_{1j}\eta_1 + a_{2j}\eta_2 + \cdots + a_{nj}\eta_n = 0 \quad (j = 1, 2, \dots, n).$$

The rank of the determinant $|a_{ij}|$ being p , there are $n - p$ linearly independent sets of solutions of each set of equations. If $p = n$, there is only the identically zero solution and the systems are each incompatible. In this case the corresponding sets of non-homogeneous equations

$$(8') \quad \sum_{j=1}^n a_{ij}\xi_j = \gamma_i, \quad (9') \quad \sum_{i=1}^n a_{ij}\eta_i = \delta_j$$

have each a unique solution. If $n - p = m \neq 0$ and the homogeneous sets (8) and (9) both have solutions, one must impose m linear conditions on the γ 's (or δ 's) in order that

the non-homogeneous system (8') (or 9') have at the same time a solution. The general solution of (8') is the sum of a particular solution and of the most general solution of (8), the latter depending on $n - p$ constants.

Turning now to a discussion of the most general linear differential equation of the second order,

$$(10) \quad L(u) \equiv u'' + Pu' + Qu \equiv \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = R,$$

where P , Q , and R are continuous functions in an interval A , B , we observe that if no boundary conditions are imposed, the solutions can be approximated by $n - 1$ non-homogeneous difference equations which take the form

$$(11) \quad A_i u_{i-1} + B_i u_i + C_i u_{i+1} = D_i \quad (i = 1, \dots, n-1),$$

in which the $n + 1$ variables u are the values of the function at equally spaced points in the interval. Since there are two more variables than equations, the values of two of them may generally be assigned at random; in other words, the solution depends linearly on two arbitrary constants. In order to arrive at unique results, it would be necessary to impose two more linear conditions. As examples of such conditions the following may be cited:

(a) The values of the function or its derivative at two points a , b , of the interval may be pre-assigned; or more generally two linear conditions of the form

$$(12) \quad U_i(u) \equiv \alpha_{i1}u(a) + \alpha_{i2}u'(a) + \beta_{i1}u(b) + \beta_{i2}u'(b) = \delta_i \quad (i = 1, 2),$$

may be imposed. More generally still these conditions may concern themselves with the values of u and u' at any number of points in the interval.

(b) Given the functions f_{i1} , f_{i2} , the solutions may be subject to the integral conditions

$$u_i(u) = \int_a^b [f_{i1}u(x) + f_{i2}u'(x)]dx = 0.$$

(c) There may be a combination of the types (a) and (b).

The equations (11) with two linear conditions such as (12) correspond to one of the sets (8'), (9').

To the homogeneous equations (8) corresponds the homogeneous differential equation

$$(13) \quad L(u) \equiv \frac{d^2 u}{dx^2} + P \frac{du}{dx} + Qu = 0,$$

together with two linear homogeneous equations of condition

$$(14) \quad U_i(u) \equiv \alpha_{i1}u(a) + \alpha_{i2}u'(a) + \beta_{i1}u(b) + \beta_{i2}u'(b) = 0 \quad (i = 1, 2).$$

The differential equation adjoint to (10) is

$$(15) \quad M(v) \equiv \frac{d^2 v}{dx^2} - \frac{d(Pv)}{dx} + Qv = R$$

and to (13) is

$$(16) \quad M(v) \equiv \frac{d^2 v}{dx^2} - \frac{d(Pv)}{dx} + Qv = 0.$$

Similarly we may have boundary conditions adjoint to (12) or (14). In the latter case they are

$$(17) \quad V_i(v) = 0 \quad (i = 1, 2),$$

and we have the formula

$$(18) \quad vL(u) - uM(v) = \frac{d}{dx}(Puv - Pv' + u'v).$$

The two systems (13), (14) and (16), (17) have the same number of linearly independent solutions or are both incompatible. The approximation equations of the two will bear to one another the same relation as (8) and its adjoint set (9), and the formula (18) has its analogue in a bilinear form involving the ξ 's and η 's. Precisely as in the case of the algebraic difference equations, the incompatibility of (13), (14) entails a unique solution of (10), (12) and of (13), (12) and of (10), (14). In case the system (13), (14) is compatible, a necessary and sufficient condition that there be a solution of the non-homogeneous system (10), (12) also is that certain linear equations of condition be imposed on R and the δ 's, while the general solution is the sum of a particular solution and of the most general solution of (13), (14).

Every differential equation of the second order may, by

a change of variable, be put into a form which is self-adjoint, that is, such that (10) and (15) and also (13) and (16) coincide. The type most important in application is that where the boundary conditions are also self-adjoint; (14) and (17) would in that case be identical. This corresponds in the algebraic cases (8), (9) to a symmetric array $|a_{ij}|$.

The discussion of the simplest cases of the real solutions of the self-adjoint equation together with the behaviors of their zeros forms the topic of Chapter III. The properties of solutions of the most general homogeneous equation of this type, which can be taken in the form

$$(19) \quad (pu')' + gu = 0,$$

follow closely those concerning the solutions of (1) which were written in explicit form. The well-known theorem of Sturm concerning the alternation of the zeros of two linearly independent solutions u_1, u_2 of (19) follows from the readily established formula

$$\frac{d(u_1/u_2)}{dx} = \frac{u_2 u_1' - u_1 u_2'}{u_2^2} = \frac{\text{constant}}{pu_2^2}.$$

Conditions under which this separation theorem holds for two linear combinations of u and u' such as

$$\psi_1 = \varphi_1(x)u + \varphi_2(x)u', \quad \psi_2 = \varphi_3(x)u + \varphi_4(x)u'$$

are readily obtained. Of course, corresponding theorems for the approximating algebraic equations can also be deduced; but they are not of special interest.

The solutions of (19) being continuous functionals of p and g , we should expect that by proper choice of variation in these coefficients the zeros of $u(x)$ would move closer together as in the special case of equation (1), and this is a result easily established. Denoting by u_1 the solution corresponding to coefficients p_1 and g_1 in (19) and by u_2 that corresponding to p_2 and g_2 , it follows from the identity, due to Picone,

$$\begin{aligned} \frac{d}{dx} \left[\frac{u_1}{u_2} (p_1 u_2 u_1' - p_2 u_1 u_2') \right] \\ = (g_2 - g_1) u_1^2 + (p_1 - p_2) u_1'^2 + p_2 \left(u_1' - u_2' \frac{u_1}{u_2} \right)^2, \end{aligned}$$

that, in an interval between two consecutive zeros x_1 and x_2 of u_1 , a necessary condition that there exist a solution $u_2 \neq 0$ is that at least one of the expressions $p_1 - p_2$, $g_2 - g_1$ be positive. For otherwise on integrating the identity between x_1 and x_2 , the two sides give opposite signs. Roughly speaking, it is true that if p decreases or g increases the zeros move closer together. An analogy can be found for the corresponding approximating difference equations.

By means of this comparison theorem the author easily finds upper and lower bounds for the lengths of intervals of oscillation by solving an equation with constant coefficients greater or less than p and g . For example, if $\gamma_1 = \min p > 0$, $\gamma_2 = \max g$, on solving

$$u'' + \frac{\gamma_2}{\gamma_1} u = 0,$$

it is found that if $\gamma_2 < 0$ the exponential result precludes the possibility of an oscillating solution of (19), while if $\gamma_2 > 0$, $\sin \sqrt{(\gamma_2/\gamma_1)} x$ gives a lower bound $\pi/\sqrt{\gamma_2/\gamma_1}$ for the interval between the zeros of that equation.

The most important special case which arises in applications is when $g = q + \lambda k$; that is, when (19) takes the form

$$(20) \quad (pu')' + (q + \lambda k)u = 0.$$

If p , q and k remain fixed and the parameter λ varies, the theorem of comparison just enunciated can be applied directly in case $k \geq 0$ or $k \leq 0$. In case $q \leq 0$ and k changes sign it may be applied by dividing the equation through by $|\lambda|$. The remaining case where no restrictions are put on q and k involves complex solutions and has been worked out in detail only since Bôcher's lectures were given.*

Let us now impose one condition on the solution $u(x)$ of (20); for example,

$$(21) \quad c_1 u(a) + c_2 u'(a) = 0.$$

(This may, by a change of variable, be reduced to $u(a) = 0$ or $u'(a) = 0$.) By proper variations of p , q and λk the lengths of the intervals may be decreased and the zeros of u and u' made to move in toward $x = a$. In particular, when p , q and k are held fixed, with increase of $|\lambda|$ the number of

* Cf. Haupt and Richardson cited above.

zeros of $u(x)$ in the interval a, b can be made to increase without limit. Further there is a set of values of $\lambda = \lambda_1, \lambda_2, \dots$ (characteristic numbers), which diverge to infinity, to which correspond the characteristic solutions $u(x)$ satisfying (21) and a second linear condition

$$(21') \quad c_3 u(b) + c_4 u'(b) = 0.$$

In case $k \geq 0$ the oscillation theorem for the system (20), (21), (21') will be approximately that for the special system (1), (2); but in case k has both signs the result must be so modified as to prescribe two characteristic numbers (one positive and the other negative) corresponding to which are solutions having any given number of zeros in the interval.

Besides a discussion of conditions under which the characteristic numbers of a differential equation will be real and the derivation of an oscillation theorem for a simple case of n differential equations each with n parameters, Chapter IV contains a detailed treatment of the oscillation theorem for the interesting periodic conditions

$$(22) \quad u(a) = u(b), \quad u'(a) = u'(b).$$

It may be remarked that when more general boundary conditions of the type (14) are imposed, it is customary to follow Hilbert in reducing them to a small number of normal forms and in discussing these in sequence. It may also be noted that the theorem of oscillation derived by Bôcher holds more generally for all boundary conditions belonging to the same normal form as (22). For $k \geq 0$ the theorem is precisely that which one would surmise from a slight consideration of the special equation (1) under these same conditions (22), viz.: corresponding to each even number of oscillations there are two characteristic solutions, to each odd number there are none, and to zero there is one.

Chapter V begins with a consideration of the existence and fundamental properties of the Green's function for differential equations of the n th order. Since the case $n = 2$ is typical, we may here confine ourselves to the equations already discussed. When the homogeneous system (13), (14) is incompatible, a Green's function $G(x, \xi)$ having the properties (a), (b), (c) enumerated above may be set up and it furnishes as a unique solution of (10), (14) the formula

$$(23) \quad u(x) = \int_a^b R(x) G(x, \xi) d\xi.$$

The Green's function for the adjoint system (15), (17) turns out to be $G(\xi, x)$, so that for a self-adjoint system it is symmetric in x and ξ , as we saw in the special form (7) set up for the system (1), (2). When a solution of the system (10), (12) is desired, the formula (23) must be modified by the addition of certain auxiliary terms. While no mention is made of the fact in the lectures, these theorems have an analogue in the theory of algebraic equations.

The same sort of argument as establishes (23) leads to a proof of the equivalence of the non-homogeneous differential system (10), (12) with the non-homogeneous integral equation

$$u(x) = f(x) + \int_a^b R(x)G(x, \xi)u(\xi)d\xi,$$

where $f(x)$ is the sum of certain auxiliary terms. In case the differential system is homogeneous, the term $f(x)$ is zero and (13), (14) is equivalent to the corresponding homogeneous integral equation

$$u(x) = \int_a^b R(x)G(x, \xi)u(\xi)d\xi$$

of which (6) is a special case.

In Chapter I the author proves by the method of successive approximations the usual existence theorem for the solution of the non-homogeneous equation (10) under the one-point boundary condition

$$(24) \quad u(c) = \gamma, \quad u'(c) = \gamma'$$

by showing that the series

$$v_1 + v_2 + v_3 + \cdots; \quad v_1' + v_2' + v_3' + \cdots$$

approximating to u and u' are uniformly convergent. This is done by establishing the inequality

$$\left| \begin{array}{l} v_n \\ v_n' \end{array} \right| \leq \frac{CL^{n-2}(2M)^{n-2}(x-c)^{n-2}}{(n-2)!},$$

where C is greater than $|v_1|, |v_1'|, |v_2|, |v_2'|$; M greater than $|P|, |Q|, |R|, |\gamma|, |\gamma'|$; and L greater than both unity and the length of the interval.

One of the contributions of Bôcher first appearing in these lectures is the answer to the question as to the nature of the

dependence of u on the coefficients P, Q, R , and the constants γ, γ' . He shows that the solution u is a continuous functional $F(P, Q, R, \gamma, \gamma')$, and moreover that this is true even when the three arguments P, Q, R have a finite number of discontinuities. It may be assumed that as in the case of functions the sum or product of two continuous functionals is a continuous functional as is also the integral of a functional. The reasoning of Bôcher, which is characteristically simple and elegant, then proceeds as follows. Since each v and v' is obtained by a finite number of integrations of explicit sums and products of P, Q, R, γ, γ' , and of specified continuous functions of x , it is a continuous functional of the arguments.

The finite sums $\sum_1^{n_1-1} v_n, \sum_1^{n_1-1} v_n'$ will then also be continuous functionals for any definite index n_1 . But for the functional field limited by the assumed value of the constant M , and for an index N_1 large enough, $\left| \sum_{n_1}^{\infty} v_n \right|$ and $\left| \sum_{n_1}^{\infty} v_n' \right|$ will be less than any preassigned $\epsilon > 0$ for $n_1 > N_1$. This establishes the theorem.

Returning to the topic of successive approximations at the end of Chapter V, Bôcher by an application of the Green's function fulfils the promise made in his International Congress lecture of considering a generalization of the theory of successive approximations. Taking the differential system in the form

$$(25) \quad L'(u) = L''(u) + R, \quad U_i'(u) = U_i''(u) + \delta_i,$$

where L' and L'' are homogeneous linear differential expressions of orders n and $m < n$ respectively and $U_i'(u)$ and $U_i''(u)$ are linear forms in $u(a), u'(a), \dots, u^{(n-1)}(a), u(b), u'(b), \dots, u^{(n-1)}(b)$ and assuming that the homogeneous system $L'(u) = 0, U_i'(u) = 0$ is incompatible, he finds that the series set up by the usual method of successive approximations as solutions of the successive equations

$$L'(u_m) = L''(u_{m-1}) + R, \quad U_i'(u_m) = U_i''(u_{m-1}) + \delta_i \\ (m = 1, 2, \dots)$$

may or may not converge uniformly. The discussion of this question of convergence is made to depend on the charac-

teristic values of the system

$$(26) \quad \begin{aligned} L'(u) &= \lambda[L''(u) + R''] + R', \\ U_i'(u) &= \lambda[U_i''(u) + \delta_i''] + \delta_i', \end{aligned}$$

where $R'' + R' = R$ and $\delta_i'' + \delta_i' = \delta_i$, and (26) reduces to (25) for $\lambda = 1$. If for u_0 we take the solution of $L''(u_0) + R'' = 0$, $U_i'' + \delta_i'' = 0$, the method of successive approximation gives us a power series in λ which will converge within a certain circle (finite or infinite) with $\lambda = 0$ as center and which will diverge without. To obtain a knowledge of the radius of this circle, Bôcher considers a differential system

$$(27) \quad L(u) = R, \quad U_i(u) = \gamma_i \quad (i = 1, 2, \dots, n),$$

of which (26) is a special case. The coefficients are assumed continuous in the real variable and analytic in λ for a certain Weierstrass domain. His profound grasp of the algebraic problem leads him to a consideration of the function

$$(28) \quad \frac{\begin{vmatrix} u_0 & y_1 & y_2 & \cdots & y_n \\ U_1(u_0) - \gamma_1 & U_1(y_1) & U_1(y_2) & \cdots & U_1(y_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U_n(u_0) - \gamma_n & U_n(y_1) & U_n(y_2) & \cdots & U_n(y_n) \end{vmatrix}}{\begin{vmatrix} U_1(y_1) & \cdots & U_1(y_n) \\ \vdots & \ddots & \vdots \\ U_n(y_1) & \cdots & U_n(y_n) \end{vmatrix}},$$

where u_1 is a solution of $L(u) = R$ and $y_1, \dots, y_n$ are a fundamental system of solutions for the homogeneous equation $L(u) = 0$. When λ is not a characteristic number of the homogeneous system $L(u) = 0$, $U_i(u) = 0$, in other words when the denominator is not zero, a substitution shows that (28) is a solution of (27). This solution is continuous in x and analytic in λ except for the characteristic numbers.

But Bôcher is able to proceed further. Even when the denominator of (28) vanishes, as it will for the characteristic number λ_i , it may happen that the numerator has a zero of the same order, in which case a solution of the system (27) still exists for this value of λ . For such a characteristic number both the homogeneous and non-homogeneous systems must have solutions. He had already noted above the corresponding condition of affairs in the algebraic problem, where

the solutions of the homogeneous equations (8) are also solutions of the adjoint non-homogeneous equations (9'). This algebraic condition which may be expressed by the formula $\sum \delta_j \xi_j = 0$, where ξ_j denotes any solution of (8), has its analogues in the transcendental case.

He is now ready to state the results for (26) and hence for (25). If the circle of convergence of the power series in λ is not infinite, it passes through one of the characteristic values of the homogeneous system

$$(29) \quad L'(u) = \lambda L''(u), \quad U_i'(u) = \lambda U_i''(u)$$

corresponding to (26). In the ordinary case this characteristic number will be that of smallest absolute value; but if it is not, each solution of (29) corresponding to a λ of smaller absolute value must have index and multiplicity equal, that is, both the homogeneous and non-homogeneous equations have solutions; and at least one solution corresponding to a point on the circle must have them unequal. If the circle of convergence has a radius greater than unity, the approximating series gives a solution of (25). For one-point boundary conditions, such as (24), there are no characteristic numbers and the series converges everywhere.

It may be remarked that the ordinary case of the theorem just given might be conjectured from a consideration of the problem from the standpoint of the integral equation

$$(30) \quad u(x) = f(x) + \lambda \int_a^b K(x, \xi) u(\xi) d\xi$$

equivalent to (26). By successive substitution of this equation into itself we get the power series

$$u(x) = f(x) + \lambda \int_a^b K(x, \xi) f(\xi) d\xi \\ + \lambda^2 \int_a^b K(x, \xi) \int_a^b K(\xi, \xi_1) f(\xi_1) d\xi_1 d\xi + \dots$$

This is the well-known Neumann-Liouville series solution of the integral equation (30), provided it converges uniformly, as it certainly will inside of any circle extending out to the nearest characteristic number of the corresponding homogeneous integral equation. This power series in λ must be identical with that obtained from (26) by successive approximations.

R. G. D. RICHARDSON

DICKSON'S HISTORY OF THE THEORY OF
NUMBERS.

History of the Theory of Numbers. By LEONARD EUGENE DICKSON. Volume I. Carnegie Institution of Washington, 1919. 486 pp.

In these days when "pure" science is looked upon with impatience, or at best with good-humored indulgence, the appearance of such a book as this will be greeted with joy by those of us who still believe in mathematics for mathematics' sake, as we do in art for art's sake or for music for the sake of music. For those who can see no use or importance in any studies in mathematics which do not smack of the machine shop or of the artillery field it may be worth while to glance through the index of authors in this volume and note the frequent appearance of names of men whose work in the "practical applications" of mathematics would almost qualify them for a place on the faculty of our most advanced educational institutions. One of the most assiduous students of the theory of numbers was Euler, whose work otherwise was of sufficient importance to attract the notice of a king of Prussia. The devotee of this peculiar branch of science can reassure himself that he is in very good company when he reads the list of authors cited in connection with the famous theorems of Fermat and Wilson in Chapter III; Cauchy, Cayley, d'Alembert, Dedekind, Euler, Gauss, Jacobi, Kronecker, Lagrange, Laplace, Legendre, Leibniz, Steiner, Sylvester, Von Staudt and a host of others, great and small, living and dead, to the number of over two hundred, who have found these absolutely "useless" theorems worthy of their most serious attention. The reviewer is firm in the faith that no great headway will ever be made in any science, least of all in mathematics, by those who are always looking for the penny. He takes comfort also in the fact that great teachers are not found among those who are scornful of mathematics for mathematics' sake. Their race is not likely to be perpetuated, and the chances are that the study of the theory of numbers will become increasingly popular as the years go by.

One is struck in glancing through the book by the remarkable combination of superstition, fancy, scientific curiosity, and patient, plodding experiment that has figured in advancing the science of the theory of numbers. Thus, in the first chapter, which has to do with the theory of perfect numbers, the first name that appears is that of Euclid, who proved that if $p = 1 + 2 + 2^2 + \dots + 2^n$ is a prime then $2^n p$ is a perfect number, that is, a number which is equal to the sum of its aliquot parts. This solid contribution of Euclid's is followed by fanciful speculations regarding the ethical import of such numbers! "Alcuin of York and Tours explained the occurrence of the number 6 in the creation of the universe on the ground that 6 is a perfect number. The second origin of the human race arose from the deficient number 8. Indeed, in Noah's ark there were eight souls from which sprung the entire human race, showing that the second origin was more imperfect than the first, which was made according to the number 6."

The requirement that p be a prime was soon overlooked by such mystics and we have a long series of writers who state that perfect numbers always end alternately in 6 and 8, and that between any two successive powers of 10 one such number is always to be found. These errors, which would have been discovered by a little patient experiment, persist among writers on the subject till the days of experimental workers like Cataldi, who noted that the fifth and sixth perfect numbers both end in 6. To get this result it was necessary for him to show that 8191 and 131071 were both primes, which he did by the straightforward method of trying as divisors every prime less than their respective square roots. Thus early in the theory arises the fundamental problem; to discover the prime factors of a given number. It was the discovery of a method for factoring such numbers as $a^n \pm 1$ that gave Fermat such power in the investigations concerning perfect numbers, so that, for example, he was able to state that there is no perfect number of 20 or of 21 digits, contrary to the opinion of those who believed that there was a perfect number between any two powers of 10. The efforts of mathematicians since Fermat have resulted in identifying twelve perfect numbers of the type $2^{n-1}(2^n - 1)$ corresponding to the following twelve values of n : 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127. The work still goes on. $2^{47} - 1$ was proved

composite by Lucas in 1876. The actual factors were not found, however, till 1903, when Professor Cole found them to be 193707721 and 761838257287. The existence of odd perfect numbers has not been as yet proved or disproved.

Chapter II gives the history of the formulas for the number and sum of the divisors of a number, together with the problems proposed by Fermat (*a*) to find a cube which when increased by the sum of its aliquot parts becomes a square, and (*b*) to find a square which when increased by its aliquot parts becomes a cube. A third problem, due to Wallace, is also treated: to find two squares, other than 16 and 25, such that if each is increased by the sum of its aliquot parts the resulting sums are equal. Among the contributors to the history of these matters we note Cardan, Mersenne, Newton, Waring, Descartes, Euler, Kronecker, and others.

Chapter III is devoted to Fermat's and Wilson's theorems, their converses and their generalizations, together with theorems on the symmetric functions of $1, 2, 3, \dots, p-1$ modulo p . The chapter begins with the astonishing statement that the Chinese seem to have known as early as 500 B.C. that $2^n - 2$ is divisible by n (n a prime). This important theorem, rediscovered some two thousand years later by Fermat, has been the center of an immense amount of activity, beginning with Leibniz who left a proof of it in manuscript. Whether, as Mahnke, who made a careful study of the Leibniz manuscripts seems to believe, Leibniz discovered the theorem independently, before he became acquainted in 1681-2 with Fermat's *Varia Opera* of 1679, or whether he heard of the theorem when he was in Paris in 1672 or when he was in London in 1673 is a question worth study. There would seem to be no reason to doubt, however, that the manuscript proof of Leibniz is the earliest known, and that to Leibniz also belongs the discovery as early as 1682 of the theorem known as Wilson's theorem, the first published proof of which was given by Lagrange, nearly a century later.

The function $a^{n-1} - 1$ which Fermat found to be always divisible by n is sometimes divisible by n^2 , as for example when $n = 11$ and $a = 3$. The question as to when this phenomenon appears was raised by Abel, answered with numerical examples by Jacobi, and studied by Eisenstein, Sylvester and many others. The question has a bearing on Fermat's last theorem, as is shown by Wieferich's theorem that if $x^n + y^n + z^n = 0$

is satisfied by integers x, y, z , prime to n (n an odd prime) then $2^{n-1} - 1$ is divisible by n^2 . Chapter IV gives the literature on this subject.

Chapter V is another long one on Euler's φ -function and generalizations of it, together with the related theory of Farey series. Few other functions of analysis have been studied with such enthusiasm by mathematicians of the first rank, and the many remarkable applications of it to other branches of analysis and to geometry are well indicated in this chapter. Euler's function has had an amusing history as to its name and the notation for it. Euler himself gave it no name and at first no notation, later using the notation $\pi(n)$ to denote the number of integers less than n and prime to it. With this definition, of course, $\pi(1) = 0$. If we define the function as the number of integers not greater than n and having with n the greatest common divisor 1 then $\pi(1) = 1$ and this value of the function of unity fits most easily in with formulas connected with it. Gauss introduced the symbol $\varphi(n)$, which seems to have a good chance of becoming permanent. Prouhet proposed the name indicator and the notation $i(n)$, which has had some adherents among French writers in spite of the fact that the name was already pre-empted for another function by Cauchy. Sylvester named the function the totient function and denoted it by $\tau(n)$. This name with Gauss' notation has been very extensively used among American writers. The literature of this function is so great that students and teachers will find this chapter very valuable, as also the shorter Chapter VI on periodic decimal fractions. An immense amount of work is necessary to discover what others have done in these fields.

Among the by-products in the study of expressions of the form $a^n - 1$, which is itself a by-product of the theory of perfect numbers, are the theories of primitive roots, binomial congruences, more general congruences, Galois imaginaries, and periodic decimal fractions. These matters are treated in Chapters VI, VII, VIII. Chapter VI, on periodic decimal fractions, and Chapter XX on properties of the digits of numbers, as well as much of Chapter XI, come under what might perhaps be called the metrical theory of numbers, having to do with properties which depend on the base employed to represent them.

The theory of the divisibility of factorial expressions

seems to have been studied for the first time by Leibniz, who noted in his manuscript that the multinomial coefficients except the first appearing in the expansion of $(a + b + c + \dots)^n$, where n is a prime, are divisible by n . Legendre followed with a formula for the highest power of a prime to be found in $m!$. The later contributions to this subject are found in Chapter IX.

One of the most interesting of Euler's many discoveries is the formula relating to $\sigma(n)$, the sum of the divisors of n . The values of this function for $n = 1, 2, 3, 4, 5, 6, 7$ are 1, 3, 4, 7, 6, 12, 8 and no one but an Euler would have been able to find any simple law connecting such an irregular series of numbers. It is interesting to find him with his eye always turned toward the fundamental problem of finding the factors of numbers. Even this formula he uses to prove that 101 is a prime. He finds that his formula gives him 102 for the sum, whence he concludes that 101 is a prime. This method should be, but is not, listed among the tests for primality in Chapter XVIII. As it stands it is of no practical use for that problem, but neither is Wilson's theorem, for that matter. In any event it is remarkable that the primality of a number should be made to depend on the sums of the divisors of a certain set of smaller numbers. Chapter X is particularly valuable as giving a list of formulas, scattered through many journals, some of which are given by one author without proof and proved or disproved by others. It is difficult for the ordinary worker to run them down. The theory of partitions, which has such important connection with this subject, is to be given a chapter in Volume II.

Chapter XI is a list of miscellaneous theorems on divisibility and theorems on the greatest common divisor and the least common multiple. Here is indeed a mixture of important work like the results of Cesàro, Gegenbauer, de la Vallée Poussin, Landau, Dedekind and Kronecker, side by side with the amusing note that the consecutive numbers, 242, 243, 244, 245 have each a square factor greater than unity. Here are found also many approximative or asymptotic formulas such as $6x/\pi^2$ for the number of integers not greater than x and divisible by no square greater than unity, the error being less than the square root of x . The history of the familiar rule for casting out 9's and 11's is reserved for Chapter XII, which after giving a faithful account of this

matter winds up with a list of over two pages of titles not reported on.

The history of factor tables and lists of primes given in Chapter XIII begins again in the remote days of Eratosthenes. The net result of over twenty centuries of labor seems to be that the list of primes up to ten million is determined with a high degree of accuracy, thanks to the work of men drawn from many different races and nations. Some day, perhaps, a machine may be constructed to extend the list further, but the methods used for computing the tables already in existence will hardly serve for higher limits. It should be clearly understood by anyone who contemplates further work in this field that the most troublesome and tedious part of the work lies not so much in the computation as in the actual printing and proof-reading of the results.

There is perhaps no problem in the theory of numbers more fascinating to the scientist, or wider in its appeal to all sorts and conditions of men, than the problem of finding a method for factoring numbers which shall be better than the straightforward one of trying as divisors the primes less than the square root. The invention of a new instrument for studying the heavens must appeal in much the same way to the astronomer. One can easily imagine the delight which Fermat must have taken in his method of factoring a number by expressing it as the difference of two squares. It is one of the few discoveries of the illustrious Frenchman which he condescends to describe in detail. He later improved on this method and found others much more powerful for dealing with numbers of certain forms. Euler was also an indefatigable worker in this direction. He showed by some six pages of calculations that the number 1000081 is expressible in only one way as the sum of two squares and so must be a prime. "*Dolendum autem est*" he mourns, "*hanc methodum non ad omnes numeros explorandos adhiberi posse.*" He proceeded to extend the method to representation by means of other binary quadratic forms, thus developing the most powerful tool now in our possession for this purpose. Legendre supplemented this method by employing the continued fraction for finding representations of the number. Gauss invented a method of exclusion, and Seelhoff made tables for the same purpose. Many other methods have been proposed, some of which are very successful if the factors are related in particular ways.

An account of all these methods is given in Chapter XIV, while in Chapters XV and XVI appear methods applicable to numbers of special forms like $2^{2^n} + 1$ and $a^n \pm b^n$.

In the early years of the thirteenth century Leonardo Pisano noticed the series of numbers 1, 2, 3, 5, 8, 13, $\dots$, each of which is the sum of the two preceding it. The study of this series and of others like it has yielded many important results. Lucas has hit upon a general theory which includes these series. The whole theory is connected with the theory of linear difference equations with constant coefficients and also with the theory of continued fractions. Chapter XVII gives the history of this important subject.

The existence of an infinitude of primes has been known since Euclid. That there are an infinite number of them in any arithmetical progression $mx + n$ where m and n are relatively prime was not proved until 1837, when Dirichlet established it by a very difficult analysis. Dirichlet also found that any primitive binary quadratic form can represent an infinitude of primes. Much important work centers in these great discoveries. Of equal difficulty and importance is the problem of finding either exactly or by an approximative formula the number of primes between given limits. The history of these problems is contained in Chapter XVIII together with the literature connected with Goldbach's conjecture, still neither proved or disproved, that every even number is the sum of two primes. Bertrand's postulate, proved by Tchebycheff, that there is at least one prime between x and $2x - 2$ for x greater than 3 is also treated in this chapter. One can not fail to be impressed with the immense fields of analysis drawn upon in the attacks on these problems.

Chapter XIX gives the history of the function $\mu(n)$ of Moebius which is useful in the inversion of series, and plays an important rôle in the derivation of many approximative formulas.

The last chapter (XX) is devoted to the listing of many curious and amusing properties of numbers, properties chiefly connected with the representation to the base 10. It is hardly likely that any important results will flow from the study of problems like finding numbers like $512 = (5 + 1 + 2)^3$ but such little things serve sometimes to attract students to more serious things.

It may, perhaps, be objected that the book is not so much

a history as a list of references from which a history of the theory of numbers might be written. Be that as it may, there is the greatest need for just such a piece of work to promote efficiency among the professional workers in this field and to prevent them from wasting their time on problems that have already been adequately treated, and also to suggest other problems which still defy analysis. It is to be hoped that the second volume will not be long delayed.

D. N. LEHMER.

THE CALCULUS OF PROBABILITY.

Calcolo delle Probabilità. By GUIDO CASTELNUOVO. Albrighi, Segati & C., Milano-Roma-Nàpoli, 1919. xxiii + 373 pp.

THE increased interest in the calculus of probability which has arisen during the past fifty years has been due in no small part to the brilliant applications of it in the field of physical phenomena. One of the most important of these, first in point of time and a model for the others, is due to Maxwell and has to do with the distribution of velocities of the molecules of a gas. Some of these investigations of physical phenomena on the basis of the laws of probability, operating under an assumed absence of determining physical laws among certain groups of phenomena, have been so successful in accounting for or predicting physical events that the conception has arisen in some quarters of the "laws of nature" as merely certain statements of average among fortuitous occurrences. It is almost uncanny to find relatively constant results of measurements of certain sorts predicted by a mathematical analysis based essentially on an assumption of chance distribution; and yet this is found in not a few important investigations.

A paradoxical situation of this sort will always excite interest. The human mind is peculiarly uncomfortable in the presence of a demonstrated result and an intuitive feeling between which there seems to be disharmony. A disturbance of our equilibrium is produced when we see the theory of probability thus accounting for what seemed to be fixed relations among phenomena. Where there is lack of equilibrium there is

restless thought; and in the presence of the latter new analyses are made of the established results of science relevant to the questions raised. This perhaps accounts in some measure for the large number of treatises on the theory of probability published in recent years.

It is interesting to observe that in the field where these physical applications were first made and where they have been most successful perhaps in relating physical phenomena by means of theoretical investigations there is yet a feeling of dissatisfaction with the situation from the critical point of view of rigorous thinking. A minute examination of the various demonstrations proposed for a justification of Maxwell's law of distribution of velocities brings out the fact that the reasoning, though meriting the attention which it has received and in its first aspects appearing persuasive if not conclusive, is nevertheless not immune to criticism. In fact there is sufficient doubt concerning the logical processes, in any of the several proofs advanced, to leave one with a desire for some additional support—so much so, that a frank and conscientious statement would probably bring out the confession that one is somewhat relieved by the fact of the success of the theory in accounting for physical phenomena.

This most noteworthy instance which arises in the applications may serve to emphasize the oft-mentioned fact that great logical difficulties are present in the whole theory of probability. It manifests itself from the outset, being present to start with in the definition of probability. The difficulty has so persisted here, in spite of all the careful analysis which has been made and the varying form of definition which has been proposed, that one cannot yet avoid a feeling of discomfort from the start. In a range of ideas having no contact with other things and no applications in other fields, such a feeling of discomfort would do much to inhibit a detailed development of theory; one would be disposed to labor longer to clear up the initial logical difficulties. But in this case there are so many points of contact with other interests and during the past fifty years such an increasing suggestiveness of fundamental value in physical investigations that one is disposed to lay aside temporarily his feeling of discomfort and to continue to proceed tentatively with the theory till he is able to check it up through a testing of the results.

When one proceeds thus in a range of ideas somewhat

lacking in clarity and in logical precision, it is peculiarly important that the significance of the initial ideas, the character of the difficulties, and the steps of the argument, shall be rendered as clear as possible through a suitable exposition. The author of the present treatise is particularly successful in making clear at the beginning of each section the problem which is to be treated in that section and then in summing up his results clearly at the proper place. Moreover the more difficult or more elusive notions are treated with care and the learner is not only presented with a correct statement but is also warned against certain probable errors. Illustrative cases are well chosen for bringing out clearly the character of the notions in consideration. If all exposition maintained in this respect the same felicity as this book, students of mathematics would be saved a great deal of energy not infrequently diverted from a mastery of the matter by the necessity of an undue effort to follow the author's form or order of presentation.

Several subjects of mathematics, depending on a minimum of technical information and detail though requiring maturity for their mastery (like the calculus of probability), would become accessible to a wider range of readers interested in science if presented in such way as to be more readily understood. Such persons, if they attempted to read the current expositions, would now too often be repelled by the forbidding style of a condensed exposition, frequent in technical mathematical works, which wastes much more energy and of greater value than that which would be expended in printing the somewhat larger volumes needed for an exposition more carefully serving the reader's comfort. It is poor economy when a little printer's ink and paper are saved at the expense of an increase in the expenditure of scientific energy needed to read an exposition which has been too much condensed through improper omissions.

Two means are used in the present work to keep the main body of the book accessible to readers with a minimum mathematical training: a considerable part of the more technical matter is put in the appendix of sixty-four pages; a few articles are marked with an asterisk and these may be omitted without destroying continuity. The latter sometimes give new results; at other times they put on a more rigorous or more exact basis what had already been developed in a more intuitive way and

with less care for completeness. An instance of the latter sort is afforded by sections 45 and 46, where one has a more satisfactory derivation and discussion of certain approximations earlier gotten in a somewhat formal way. The less mature reader may omit these more critical sections; he will perhaps not be dissatisfied with the earlier work, where approximate formulas are used rather too much as exact formulas are employed.

About half of the nearly three hundred pages in the main body of the work is devoted to the more elementary and more general aspects of the theory. Here the exposition is particularly marked by that clarity of expression which is characteristic of the entire work. The reader is led in a pleasing way through such topics as probability and frequency, total and composite probability, mathematical expectation and mean value, the problem of repeated trials and the theorem of Bernoulli, the normal law of probability, and probability involving continuous distribution. Then one meets in the second half of the text brief chapters on probability of causes, statistical interpretation, errors of observation, method of least squares, and the kinetic theory of gases. The greater portion of the appendix is given to a careful consideration of the theorem of Laplace-Tchebychef and an extension of it.

R. D. CARMICHAEL.

CORRECTIONS.

THE following errata have come to the attention of the editors of the BULLETIN:

In Professor Brouwer's paper: "Intuitionism and formalism," in volume 20, pages 81-96,

page 91, in the footnote, *for*: the proof *read*: the formalistic proof of this property; and interchange "können wir bestimmen" with "muss es geben;"

page 94, line 4, *for*: but that it is impossible *read*: and when a proof is added (for the rest not recognized by the intuitionist) that it is impossible.

In Professor Forsyth's paper: "Formulas for constructing abridged mortality tables for decennial ages," in the current volume,

page 35, formula (1), w_0 should be w_0/t ;
 page 37, formula (5i'), the last + sign should be -;
 page 38, formula (5t), the coefficient $6\frac{1}{2}$ should be $5\frac{1}{2}$;
 page 38, formula (5t'), the last + sign should be -.

In Professor Hayashi's paper: "On the rectifiability of a twisted cubic," in the current volume,

page 74, line 8, for $xy\bar{z}$ read: $|xy\bar{z}|$.

In Professor Carmichael's paper: "Note on a physical interpretation of Stieltjes integrals," in the current volume, pages 102-105, the last paragraph of the paper should be deleted.

NOTES.

THE twenty-sixth annual meeting of the American Mathematical Society will be held in New York City on Tuesday and Wednesday, December 30-31, 1919, immediately preceding the meeting of the Mathematical Association of America. The annual election of officers of the Society will close on Wednesday morning. The joint dinner of the two organizations will be held on Wednesday evening. Titles of papers for this meeting should be in the hands of the Secretary by December 9.

THE thirteenth western meeting of the Society, combining the meetings of the Chicago and the Southwestern Sections, will be held at St. Louis in affiliation with the American Association for the Advancement of Science on December 30-31. Retiring addresses will be delivered by Professor G. D. BIRKHOFF, as chairman of Section A, and Professor G. A. BLISS, as chairman of the Chicago Section. Titles and abstracts of papers should be in the hands of Professor ARNOLD DRESDEN, secretary of the Chicago Section, by December 2.

A NEW edition of the List of Officers and Members of the Society is in preparation and will be issued in January next. Blanks for furnishing information have been sent to the members, and prompt response is requested in order that the List may be as accurate as possible.

THE opening (September) number of volume 21 of the *Annals of Mathematics* contains the following papers: "Investigation of a class of fundamental inequalities in the

theory of analytic functions," by J. L. W. V. JENSEN, translated by T. H. GRONWALL; "Functions of limited variation in an infinite number of dimensions," by P. J. DANIELL; "A new sequence of integral tests for the convergence and divergence of infinite series," by R. W. BRINK; "Calculation of the complex zeros of the function $P(z)$ complementary to the incomplete gamma function," by P. FRANKLIN; "Total differentiability," by E. J. TOWNSEND.

THE following nine doctorates with mathematics as major subject were conferred by American universities in the academic year 1918-1919; the title of the dissertation is added in each case: LOUIS BRAND, Harvard, "I. On linear equations with an infinite number of variables. II. On infinite systems of linear integral equations. III. Flexual deflections and statically indeterminate beams"; G. H. CRESSE, Chicago, "On the class numbers of binary forms;" JESSIE M. JACOBS, Illinois, "The trilinear binary form as a cubic surface;" GERTRUDE I. MCCAIN, Indiana, "Series of linear iterated fractional functions—character of the functions;" C. N. REYNOLDS, Harvard, "On the zeros of solutions of linear differential equations;" WAYNE SENSENIG, Pennsylvania, "The invariant theory of involutions on conics;" CHAN-CHAN TSOO, Harvard, "The geometry of a non-euclidean line-sphere transformation;" C. L. E. WOLFE, California, "On the indeterminate cubic equation $x^3 + Dy^3 + D^2z^3 - 3Dxyz = 1$;" H. E. WOLFE, Indiana, "A study of plane circle to circle transformations by means of tetracyclic coördinates."

At the meeting of the National academy of sciences at New Haven, November 10-12, the papers read included the following: By E. W. BROWN: "Jupiter's five attendant planets;" by EDWARD KASNER: "Some new theorems in the dynamics of particles;" by J. K. WHITTEMORE: "The starting of a ship under constant power."

At the University of Strasbourg, the following courses in pure and applied mathematics are announced for the academic year 1919-1920:—By Professor FRÉCHET: Functional calculus, two hours first semester, three hours second semester; Approximating functions, one hour first semester.—By Professor VALIRON: Entire functions, two hours second

semester.—By Professor VILLAT: Rational mechanics, three hours first semester, one hour second semester; Elliptic functions with application to mathematical physics, two hours second semester.—By Professor ESCLANGON: Astronomy, two hours. Courses in pure mathematics will also be given by Professor PÉRÈS. MM. ANTOINE and DARMOIS are *maitres de conférences* in mathematics, and M. VÉRONNET is chargé de conférences in rational mechanics.

VUIBERT, of Paris, has announced that publication of the following periodicals is to be resumed this fall: *L'Education mathématique* (which last appeared in July, 1914) and *Revue de Mathématiques spéciales* (last published in September, 1914).

THE Belgian academy of sciences has awarded the Deruyts prize to Dr. LUCIEN GODEAUX, for his researches in algebraic geometry. The Academy states that an extension of time has been granted during which memoirs may be presented for two prizes, of 800 francs each, first announced March 7, 1914 for award August 1, 1915. The subjects are: The infinitesimal geometry of curved surfaces, and Systems of conics in space. Competing memoirs should be written in French, Flemish, or Latin, and should be sent to the Secretary at the Palais des Académies, Brussels, before August 1, 1920.

THE decennial prize in pure mathematics founded by the Belgian government has been awarded to Professor A. DEMOULIN, for the decade 1904–1913.

THE Paris academy of sciences announces the award of the following prizes in pure and applied mathematics: the Francœur prize (1,000 francs) to Dr. G. GIRAUD, for his contributions to the theory of automorphic functions; the Poncelet prize (2,000 francs) to Gen. P. CHARBONNIER, for his researches in ballistics; the Benjamin Valz prize (460 francs) to F. BOQUET, for the totality of his scientific work; the de Pontécoulant prize (700 francs) to Professor A. S. EDDINGTON, for his work in celestial mechanics; the Petit d'Ormoy prize (10,000 francs) to Professor H. LEBESGUE, for his contributions to mathematics. The foundation for scientific research has awarded the Gegner stipend (4,000 francs) to Professor R. BAIRE, for his work in the general theory of functions.

DR. J. FAIRON has been appointed professor of mathematics at the University of Liège.

PROFESSORS E. L. A. MERLIN and M. L. M. STUYVAERT have been promoted to full professorships of mathematics at the University of Ghent.

At the University of Nancy, Dr. J. HAAG has been appointed professor of the calculus, and Professors GOT, of the Lycée at Marseilles, and LEAU, of the Lycée St. Louis, are in charge of courses. Professor A. M. G. FLOQUET has retired from active teaching.

ASSOCIATE professor E. GAU has been appointed to the professorship of infinitesimal analysis at the University of Grenoble, succeeding Professor COLLET, who has retired from active teaching.

At French universities, the following *maîtres de conférences* have been appointed: MM. CHAPELON and KAMPÉ de FÉRIET, at the University of Lille; JANET, at the University of Grenoble; TURRIÈRE, at the University of Montpellier.

PROFESSOR V. J. BOUSSINESQ, of the University of Paris, has retired from active teaching.

DR. G. HESSENBERG has been appointed professor of mathematics at the University of Tübingen.

PROFESSOR W. KILLING, of the University of Munich, has retired from active teaching.

DR. F. SCHUR has been appointed to a professorship at the University of Breslau.

DR. H. TIEZTZE has been appointed professor of mathematics at the University of Erlangen.

At the University of Freiburg i. Br., Dr. A. LOEWY has been appointed professor of mathematics, and Professor L. STICKELBERGER has retired from active teaching.

THE library of the late Professor MAXIME BÔCHER, of Harvard, has been purchased for the Aberdeen Proving Ground.

THE mathematical library of the late Dr. R. A. HARRIS has been presented to Cornell University by Mrs. Harris. The collection comprises nearly three thousand volumes, including practically the complete literature on the theory of tides. It will be housed in the mathematical seminar.

AT the newly established southern division of the University of California, located at Los Angeles, the normal school and collegiate mathematics are the only branches of mathematics as yet provided. Miss MYRTIE COLLIER has been made head of the normal department, and Mr. G. E. F. SHERWOOD will have charge of the collegiate mathematics.

THE honorary degree of doctor of laws was conferred on Professor T. F. HOLGATE, of Northwestern University, by Queen's University, Ontario, in October.

ASSISTANT professor W. D. MACMILLAN, of the University of Chicago, has been promoted to an associate professorship of astronomy.

PROFESSOR A. A. BENNETT, of the University of Texas, has been granted a second year's leave of absence to serve as mathematics and dynamics expert in the ordnance bureau of the war department at Washington.

ASSISTANT professor H. N. DAVIS, of Harvard University, has been promoted to a full professorship of mechanical engineering.

AT the University of Michigan, the following changes have been made in the mathematical staff: associate professors PETER FIELD and L. C. KARPINSKI have been promoted to full professorships, and assistant professor J. W. BRADSHAW to an associate professorship; Dr. E. S. ALLEN has resigned to accept an assistant professorship at the University of West Virginia; Mr. E. C. BLANKENSTEIN, Mr. M. F. JOHNSON, Mr. D. C. KUZORINOFF, and Mr. O. J. PETERSON have been appointed instructors.

ASSISTANT professor W. E. MILNE, of the University of Oregon, has been promoted to a full professorship of mathematics.

AT the centennial celebration of Colgate University on October 11, JAMES M. TAYLOR, professor of mathematics in the University since 1870, received the honorary degree of doctor of science.

IN the mathematics department of the University of Minnesota, the following changes have been made: assistant professor R. R. SHUMWAY has been promoted to an associate professorship; Dr. C. H. YEATON has resigned to become professor of mathematics at the School of Engineering of Milwaukee; Miss OLIVE ATWOOD and Mr. R. M. MATHEWS have been appointed instructors.

PROFESSOR F. J. HOLDER, of the University of Pittsburgh, has been appointed professor of mathematics and dean of the school of commerce at Mercer University, Macon, Ga.

AT Vassar College, assistant professor LOUISE D. CUMMINGS has been promoted to an associate professorship and Dr. MARY E. WELLS to an assistant professorship of mathematics.

ASSOCIATE professor R. P. STEPHENS, of the University of Georgia, has been promoted to a full professorship of mathematics.

AT the Armour Institute of Technology, assistant professor W. C. KRATHWOHL has been promoted to an associate professorship and Dr. W. L. MISER, of the University of Arizona, has been appointed assistant professor of mathematics.

AT the University of California, associate professor C. A. NOBLE has been promoted to a full professorship of mathematics.

AT Washington University, St. Louis, assistant professor OTTO DUNKEL has been promoted to an associate professorship, and Dr. J. R. MUSSELMAN has been appointed instructor in mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- ARWIN (A.). Ueber das Auflösen der Kongruenzen von dem dritten und vierten Grade nach einem Primzahlmodulus. Leipzig, O. Harrassowitz, 1916.
- BARRAU (J. A.). Analytische meetkunde. Eerste deel: Het plat vlak. Groningen, Noordhoff, 1918. 8vo. 408 pp. Fl. 8.50
- BIRKELAND (R.). Lærebok i matematisk analyse, differential- og integralregning. Differentialligninger. Trondhjem, F. Bruns, 1917. 8vo. 12 + 500 pp. Kr. 22.50
- BOOLE (M. E.). Philosophy and fun of algebra. London, C. W. Daniel, 1918. Cr. 8vo. 3s.
- CRATHORNE (A. R.). See RIETZ (H. L.).
- CZUBER (E.). Vorlesungen über Differential- und Integralrechnung. 2ter Band. 4te Auflage. Leipzig, Teubner, 1919. 12 + 599 pp.
- DAVISON (C.). Differential calculus for colleges and secondary schools. London, Bell, 1919. 8 + 309 pp. 6s.
- DECKERT (A.). Einführung in die Vektor-Rechnung. (Sammlung Kösel.) Kempten und München, J. Kösel, 1916. 8vo. 4 + 84 pp. M. 1.95
- FLEISCHMANN (K.). Die geodätischen Linien auf Rotationsflächen. (Diss.) Breslau, 1915.
- HENRY (V.). Das erkenntnistheoretische Raumproblem in seinem gegenwärtigen Stande. Berlin, Reuther und Reichard, 1915. 8vo. 98 pp. M. 3.20
- OTTO (F. A.). Herausforderung an alle Mathematiker der Welt oder die Lösung der Fermatschen Probleme. Berlin, C. Kroll, 1918. 8vo. 96 pp. M. 5.00
- PINCHERLE (S.). Algebra complementare. 1a parte: Analisi algebrica. 3a edizione. (Manuali Hoepli, No. 141.) Milano, Hoepli, 1917. 8vo. 174 pp. L. 1.50
- RIETZ (H. L.). and CRATHORNE (A. R.). College algebra. Revised edition. New York; Holt, 1919. 8vo. 12 + 268 pp.
- SILBERSTEIN (L.). Projective vector algebra: an algebra of vectors independent of the axioms of congruence and of parallels. London, Bell, 1919. 7 + 78 pp. 7s. 6d.
- STONEY (J.). An introduction to the differential and integral calculus, for the use of engineering and technical students. London, Pitman, 1918. Cr. 8vo. 146 pp. 3s. 6d.

II. ELEMENTARY MATHEMATICS.

- AHRENS (W.). Mathematische Unterhaltungen und Spiele. 2ter Band. 2te Auflage. Leipzig, Teubner, 1918. 455 pp. Geb. M. 15.00
- BETZ (W.). Geometry for junior high schools. (Elements of mathematics, book 1.) Rochester, N. Y., Board of Education, 1919. 12mo. 16 + 111 pp.

———. Introductory algebra exercises. Rochester, N. Y., Board of Education, 1918. 12mo. 6 + 73 pp.

DENNIS (T.). An arithmetic for preparatory schools, with answers. 2d edition. London, Bell, 1919. 14 + 376 pp. 4s. 6d.

JUNIOR geometry for schools. 5th edition. Dublin, Thom, 1918. 186 pp. 2s.

MILNE (R. M.). Mathematical papers for admission into the Royal military academy and the Royal military college and papers in elementary engineering for naval cadetships for the years 1909-1918. Edited by R. M. Milne. London, Macmillan, 1919. 7s.

III. APPLIED MATHEMATICS.

ALGER (P. L.) and INGALLS (J. M.). Groundwork of practical naval gunnery, or exterior ballistics. Annapolis, U. S. Naval Institute, 1918. 4to. 360 pp. \$4.50

AUERBACH (F.). Grundbegriffe der modernen Naturlehre. (Einführung in die Physik.) 4te Auflage. (Aus Natur und Geisteswelt, Nr. 40.) Leipzig, Teubner, 1917. 146 pp. Geb. M. 1.90

BATES (E. L.) and CHARLESWORTH (F.). Practical mathematics. 2d edition. London, Batsford, 1918. Cr. 8vo. 5s.

BECQ (A.). Electricité théorique et industrielle. Livre 1: Electricité théorique. Livre 2: Machines électriques. 8e édition. Paris, Librairie de l'Ecole spéciale des Travaux publics, 1918. 8vo. 172 + 155 pp. Fr. 8.00 + 12.00

BRYK (O. J.). See KEPLER (J.).

CARD (S. F.). Air navigation: notes and examples. 3d edition. London, Arnold, 1919. 6 + 140 pp. 10s. 6d.

CHARLESWORTH (F.). See BATES (E. L.).

CYCLOPEDIA of drawing: a practical reference work on mechanical and architectural drawing; prepared by a staff of engineers, architects, designers, . . . 3d edition. Chicago, American Technical Society, 1919. 8vo. 1620 pp. 4 vols. \$13.80

DANDER (M.). Dictionnaire international de navigation aérienne et constructions aéronautiques (français, italien, anglais, allemand). Milano, Hoepli, 1919. 16mo. 227 pp. L. 6.50

EINSTEIN (A.). Ueber die spezielle und die allgemeine Relativitätstheorie, gemeinverständlich. (Sammlung Vieweg, Tagesfragen aus den Gebieten der Naturwissenschaften und der Technik.) Braunschweig, Vieweg, 1917. 8vo. 4 + 76 pp. Geh. M. 2.80

FRY (H. P.). Notes on mechanical drawing. 6th edition. Philadelphia, H. P. Fry, 1919. 8vo. 85 pp. \$1.50

HEUVELINK (H. J.). De stereografische kaartprojectie in hare toepassing bij de rijksdriehoeksmeting. Delft, J. Waltman, Jr., 1918. 8vo. 48 pp.

HICKS (J. W.). The theory of the rifle and rifle shooting. London, Charles Griffin, 1919. 129 pp. 5s.

HILDEBRANDSSON (H. H.). Samuel Klingenskiernas levnad och verk. Biografisk skildring utgiven av K. Svenska Vetenskapsakademien. I: Levnadsteckning. Stockholm, Almqvist & Wiksells Boktryckeri, 1919. 88 pp.

- HOHENNER (H.). Der Hohennersche Präzisionsdistanzmesser und seine Verbindung mit einem Theodolit. Leipzig, Teubner, 1919. 64 pp. Geh. M. 3.20
- HOSMER (G. L.). Geodesy. Including astronomical observations, gravity measurements and method of least squares. New York, Wiley, 1919. 11 + 368 pp. \$3.50
- INGALLS (J. M.). See ALGER (P. L.).
- JACOBY (H.). Navigation. 2d edition. New York, Macmillan, 1919. 11 + 350 pp.
- JÄGER (G.). Theoretische Physik. I: Mechanik und Akustik. 4te verbesserte Auflage. (Sammlung Götschen, Nr. 76.) Berlin, Götschen, 1916. 16mo. 167 pp. Geb. M. 1.25
- KEPLER (J.). Die Zusammenklänge der Welten. Neue Sternkunde. Auseinandersetzung mit dem Sternenherold. Schöpfungsgeheimnis in Weltentiefen. Herausgegeben von O. J. Bryk. Jena, Diederichs, 1918. 368 pp. Geh. M. 12.00
- LEPPER (G. H.). From nebula to nebula or the dynamics of the heavens. 4th edition, revised and enlarged. Pittsburgh, G. H. Lepper, 1919. 401 pp.
- LOCHE (L. E.). Des mécanismes élémentaires. Paris, 1919. 8vo. 12 + 257 pp. Fr. 12.00
- LOPEZ SOLER (J.). Gráfico de calculos JuLius. Madrid, 1918.
- OBERMILLER (J.). Der Kreislauf der Energien in Natur, Leben und Technik. Leipzig, Barth, 1919. Geb. M. 3.60
- STRINGFELLOW (G. W.). Industrial mathematics. London, Pitman, 1918. 8vo. 124 pp. 1s. 6d.
- WIEN (W.). Neuere Entwicklung der Physik und ihrer Anwendungen. Leipzig, Barth, 1919. Geb. M. 6.00

Publications of the American Mathematical Society

Transactions of the American Mathematical Society.

Published quarterly. Subscription price for annual volume, \$5.00; to members of the Society, \$3.75.

Mathematical Papers of the Chicago Congress, 1893.

Price, \$3.00; to members, \$1.50.

Evanston Colloquium Lectures, 1893. By FELIX KLEIN

Price, 75 cents.

Boston Colloquium Lectures, 1903. By H. S. WHITE, F.

S. WOODS and E. B. VAN VLECK. Price, \$2.00; to members, \$1.50.

Princeton Colloquium Lectures, 1909. By G. A. BLISS

and EDWARD KASNER. Price, \$1.50; to members, \$1.00.

Madison Colloquium Lectures, 1913. By L. E. DICKSON

and W. F. OSGOOD. Price, \$2.00; to members, \$1.50.

Cambridge Colloquium Lectures, 1916. Part I. By G.

C. EVANS. Price, \$1.50; to members, \$1.00.

Address all orders to AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.

Whole No. 283

CONTENTS

	PAGE
Note on Convergence Tests for Series and on Stieltjes Integration by Parts. By Professor R. D. CARMICHAEL	97
Note on a Physical Interpretation of Stieltjes Integrals By Professor R. D. CARMICHAEL	102
A Derivation of the Equation of the Normal Probability Curve. By Professor W. D. CAIRNS	105
Bôcher's Boundary Problems for Differential Equations. By Professor R. G. D. RICHARDSON	108
Dickson's History of the Theory of Numbers. By Professor D. N. LEHMER	125
The Calculus of Probability. By Professor R. D. CARMICHAEL	132
Corrections	135
Notes	136
New Publications	142

Changes of address of members, exchanges, and subscribers should be communicated at once to the Secretary of the American Mathematical Society, 501 West 116th Street, New York.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to The American Mathematical Society, 41 North Queen St., Lancaster, Pa., or 501 West 116th Street, New York.

The initiation fees and annual dues of members of the American Mathematical Society are payable to the Treasurer of the American Mathematical Society, Professor J. H. TANNER, Cornell University, Ithaca, N. Y.

Articles for insertion in the BULLETIN should be addressed to the
BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.

The following dates have been fixed for the meetings of the Society :

Annual Meeting, New York: December 30-31, 1919

*Western Meeting, Chicago and Southwestern Sections, St. Louis,
December 30-31, 1919*

IVARD COLLE
JAN 30 1920

LIBRARY

BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY

**A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE**

EDITED BY

F. N. COLE	VIRGIL SNYDER	J. W. YOUNG
ALEXANDER ZIWET	D. E. SMITH	
T. LEVI-CIVITA	R. C. ARCHIBALD	

VOLUME XXVI., NUMBER 4

JANUARY, 1920

**PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK**

1920

Entered at the Post Office at Lancaster, Pa., as second class matter.
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER

Whole No. 284

Digitized by Google
\$5.00 a year



For senior mathematics in colleges and courses
Introductory to graduate work in
mathematics

Theory of Maxima and Minima

By HARRIS HANCOCK

University of Cincinnati

Makes available in English the important advances in this branch of mathematics that have for some time been accepted in Europe but which have not been accessible to our students in their own language. It contains numerous examples, problems, and exercises taken from algebra, geometry, physics, mechanics, and other branches of mathematics hitherto regarded as somewhat foreign to the subject.

GINN AND COMPANY

70 Fifth Avenue

NEW YORK

THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE two hundred and fifth regular meeting of the Society was held at Columbia University on Saturday, October 25, 1919. The following forty-eight members attended the two sessions:

Dr. J. W. Alexander, Professor R. C. Archibald, Dr. I. A. Barnett, Dr. Charlotte C. Barnum, Professor A. A. Bennett, Professor W. J. Berry, Professor G. D. Birkhoff, Professor Pierre Boutroux, Dr. G. A. Campbell, Professor F. N. Cole, Dr. Tobias Dantzig, Professor L. P. Eisenhart, Professor T. S. Fiske, Professor W. B. Fite, Dr. T. H. Gronwall, Professor C. O. Gunther, Professor Olive C. Hazlett, Dr. A. A. Himwich, Mr. S. A. Joffe, Professor Edward Kasner, Mr. H. P. Kean, Professor O. D. Kellogg, Professor C. J. Keyser, Dr. K. W. Lamson, Professor P. H. Linehan, Professor James Macloy, Professor R. L. Moore, Professor G. W. Mullins, Mr. George Paaswell, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor H. W. Reddick, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor D. E. Smith, Professor P. F. Smith, Dr. J. M. Stetson, Mr. J. J. Tanzola, Professor H. D. Thompson, Professor Oswald Veblen, Mr. A. C. Washburne, Mr. H. E. Webb, Professor J. H. M. Wedderburn, Professor Mary E. Wells, Mr. R. A. Wetzell, Mr. J. K. Whittemore, Professor A. H. Wilson.

Vice-President G. D. Birkhoff occupied the chair. The Council announced the election of the following persons to membership in the Society: Dr. C. C. Camp, University of Illinois; Professor C. J. Coe, University of Michigan; Dr. Teresa Cohen, Johns Hopkins University; Mr. W. E. Heal, U. S. Bureau of Plant Industry; Dr. C. A. Nelson, University of Kansas; Mr. J. L. Walsh, Harvard University. Four applications for membership in the Society were received.

A committee was appointed to audit the accounts of the Treasurer for the current year. A list of nominations for officers and other members of the Council was adopted and ordered printed on the official ballot for the annual election. Propositions for establishing a board of custodians of the property of the Society and for issuing a new catalogue of the library were laid over for action at the annual meeting.

The following papers were read at the October meeting:

- (1) Mr. J. K. WHITTEMORE: "Motion in a resisting medium."
- (2) Miss ANNA M. MULLIKIN: "A countable collection of mutually exclusive closed point sets with connected sum."
- (3) Professor OLIVE C. HAZLETT: "New proofs of certain finiteness theorems in the theory of modular covariants."
- (4) Professor R. D. CARMICHAEL: "On the convergence of certain classes of series of functions."
- (5) Professor R. D. CARMICHAEL: "Note on a transformation of series similar to the principle of inversion in the theory of numbers."
- (6) Professor R. D. CARMICHAEL: "Note on the theory of integral functions of the first class."
- (7) Professor ANNA J. PELL: "Two linear integral equations with two parameters."
- (8) Professor G. D. BIRKHOFF: "Note on stable periodic orbits."
- (9) Professor O. D. KELLOGG: "Some problems connected with submarine acoustics."
- (10) Dr. T. H. GRONWALL: "Differential variations in ballistics, with applications to the qualitative properties of the trajectory."
- (11) Professor A. A. BENNETT: "Standard density, temperature and pressure of air aloft."
- (12) Professor A. A. BENNETT: "The probable error of a small number of rounds."
- (13) Professor A. A. BENNETT: "The physical bases of ballistic table computations."
- (14) Professor A. A. BENNETT: "The sign of the distance in analytical geometry."
- (15) Dr. E. F. SIMONDS: "Invariants of infinite groups in the plane."
- (16) Professor EDWARD KASNER: "The motion of n bodies, under any forces, starting from rest."

The paper of Miss Mullikin was communicated to the Society and read by Professor R. L. Moore. Dr. Simonds' paper was read by Professor Kasner. The papers of Professor Carmichael and the last three papers of Professor Bennett were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Mr. Whittemore's paper is devoted to the study of the motion of a particle whose tangential acceleration is a given function of the velocity alone, $a = F(v)$. It is proved that if λ is a simple positive root of $F(v) = 0$ and $F(v)$ is subject to certain other conditions, generally satisfied in a real problem, $\lambda - v$ approaches zero at least as rapidly as e^{-mt} , where t is the time and m a positive constant; that under the same conditions $\lambda t - x$, where x is the distance moved, approaches a limit L , and differs from L by a quantity not greater than a constant multiple of e^{-mt} . When the root λ is not simple the results are different. Various examples are considered and some numerical illustrations given. Four problems are discussed in which there act forces working at a constant rate. Such problems seem to have been neglected in books on elementary mechanics, but have nevertheless some interest. Especial study is made of the starting of a steamer driven by engines developing constant horsepower.

2. The sum of a finite number (more than one) of mutually exclusive closed point sets is never connected.* Nor is the sum of a countably infinite number of such point sets connected if each of them consists of only one point. In one dimension no countably infinite collection of mutually exclusive closed point sets ever has a connected sum. One might rather naturally be inclined to believe that this proposition holds true also in two dimensions. Miss Mullikin shows by an example that this is, however, not the case.

3. Professor Hazlett's paper gives new proofs of certain finiteness theorems in the theory of modular invariants. In 1909 Professor Dickson proved the finiteness theorem for modular invariants in the Galois field $GF[p^n]$. This was followed in 1913 by his proof of the finiteness theorem for modular covariants. The next year, Professor Wiley proved the finiteness theorem for modular invariants of a system of forms over the Galois field $GF[p]$ and any number of cogredient points (x_i, y_i) . This proof used a lemma which is closely related to Hilbert's well known theorem on a set of polynomials.

* A set of points is said to be connected if, however it be divided into two mutually exclusive subsets, one of them contains a limit point of the other one.

In a paper presented at Chicago in December, 1918, the author proved a theorem which enables us to construct all modular covariants of a system S of forms from the modular invariants of an enlarged system S' consisting of the forms of S and an additional linear form. By using this theorem, and the finiteness theorem for modular invariants, we can prove at once the finiteness theorem for modular covariants as well as the more general theorem proved by Professor Wiley. The present proof gives a method for obtaining from a fundamental set of invariants of S' a set of modular covariants of S in terms of which all modular covariants of S are expressible as polynomials.

Finally, by means of a symbolic notation, the author proves from this theorem that every formal covariant of a system of forms over the Galois field $GF[p^n]$ is a polynomial in the modular covariants which have been made formally invariant as to the coefficients of the forms and in the irreducible covariants which are congruent to zero whenever the coefficients are marks of the field. This theorem is verified for the binary quadratic modulo 3 and for the binary cubic modulo 2.

4. In this paper Professor Carmichael develops the first fundamental convergence properties of certain very general classes of series including many of those which are classic in analysis, such as the power series, factorial series, and Dirichlet series.

5. A certain general principle of transformation of series analogous to the principle of inversion in the theory of numbers is here employed by Professor Carmichael to derive a few classic results; and the principle itself is extended and more general results are obtained through use of generalizations of the λ -function often employed in the theory of the distribution of prime numbers.

6. The object of this note by Professor Carmichael is to extend certain of the classic theorems about integral functions. It contains a modification of the Weierstrass factor theorem valid in certain cases and some consequences relative to the relation between the zeros of an integral function and those of its derivative.

7. It is shown in Mrs. Pell's paper that there exist real values of λ and μ for which the two linear integral equations

$$u(x) = \lambda \int_a^b K(x, y)u(y)dy + \mu \int_a^b L(x, y)u(y)dy,$$

$$v(s) = \lambda \int_c^d M(s, t)v(t)dt - \mu \int_c^d N(s, t)v(t)dt$$

have solutions $u(x) \not\equiv 0$ and $v(s) \not\equiv 0$, if K, L, M, N are continuous and symmetric and L, N are positive definite. Functions of two variables, which can be expressed in a certain form, can be expanded into a uniformly convergent series in terms of the functions of a system adjoint to the system $\{u_i(x)v_i(s)\}$.

8. The object of this note by Professor Birkhoff is to establish that infinitely many periodic orbits of certain types exist in the infinitesimal vicinity of any stable periodic orbit.

The note will appear in the *Proceedings of the National Academy of Sciences*.

9. Professor Kellogg outlined a number of mathematical problems which have arisen in connection with the detection of submerged submarines by sound.

10. Dr. Gronwall's paper gives a method for computing differential variations, based on the discovery of a new algebraic integral of Bliss's adjoint system of differential equations. By means of this integral, the third order system of Bliss reduces to a linear differential equation of the second order, and the numerical computation of the differential variations is materially shortened both for ordinary and anti-aircraft trajectories.

The linear differential equation referred to being of the second order, the general behavior of its solutions can be determined without much difficulty, and there result a large number of qualitative properties of the trajectory, for instance the following: the lateral deflection of the projectile due to a constant cross wind is always less than the increase in range due to a following wind of the same velocity.

11. This paper by Professor Bennett discusses the special question of the choice of certain functions assumed to represent standard density, temperature and pressure of air aloft as functions of the altitude measured in meters. The necessity for such standard functions and the reason for the exponential forms selected as against meteorological means, are explained, as applied to the problem of exterior ballistics.

12. Many known variables are taken explicitly into account to secure as great accuracy as possible in firing heavy artillery. There remains, however, an inevitable dispersion, upon which also computations are based through the magnitude of the probable errors in range and deflection. As an ordnance problem, the probable error of the various types of ammunition must be measured. A consideration of the accuracy of the probable error when based for the sake of economy on a relatively small number of rounds is here investigated by Professor Bennett, along the lines laid out by Czuber in his standard treatises *Beobachtungsfehler* and *Wahrscheinlichkeitsrechnung*. The paper will appear shortly in the *Journal of the U. S. Artillery*.

13. A pamphlet entitled "The Physical Bases of Ballistic Table Computations" is shortly to appear as an Ordnance Pamphlet, in the series "Notes on the Construction of Ordnance." This pamphlet by Professor Bennett involves no mathematical computations on the part of the reader and is rather a critical résumé of the latest practices in ballistic theory in so far as they deal with the topic mentioned. It is designed for use as a reference text in courses on gunnery, and will also constitute a part of the general introduction to the ballistic tables now being prepared under the supervision of the author.

14. In this paper Professor Bennett discusses the elementary question of the definition of distance in analytical geometry. The distance between points and between a point and a line are assumed as defined up to the algebraic sign. Three simple conventions are possible, (1) the arithmetical, by means of which all distances are positive, (2) the transcendental, by which distances are measured vectorially with a convention arbitrarily fixed in each case, (3) the algebraic, in which either

sign is always possible. Each definition has its objections, the most common usage being perhaps the least convenient. The difficulties are illustrated by very simple examples. The paper appeared in the October number of the *American Mathematical Monthly*.

15. In his Columbia dissertation, published in the *Transactions* (1918), Dr. Simonds studied the order and the number of invariants of differential configurations; and in particular with respect to the group of all point transformations obtained results entirely different from those given by Rabut in his paper on "Invariants universelles" (1898), but consistent with Kasner's invariant of the second order (*American Journal of Mathematics*, 1906).

In the present paper, Dr. Simonds discusses the five types of infinite groups of point transformations and the three types for contact transformations, according to Lie's classification, finding for each type the smallest number of curves, with distinct tangents, through a common point, which have an invariant of any given order n . For example, under the conformal group, two curves obviously have an invariant of first order, four curves have one of second order (checking with a result given by Kasner), three curves have an invariant for each of the orders 3, 4, 5, $\dots$. For each type of group the simplest invariants are given explicitly.

16. In his discussion of the general geometric properties of systems of dynamical trajectories, Professor Kasner incidentally obtained the simple theorem that any particle starting from rest in any positional field of force describes a trajectory whose initial curvature is one third the curvature of the line of force passing through the starting point (*Transactions*, 1905, Princeton Colloquium Lectures 1913, page 9). This result holds for space of any dimensionality. In the present paper the theorem is extended to the motion of any number of particles acted upon by any positional forces, conservative or non-conservative. The ratio 1:3 remains valid even in the case of a resisting medium provided the resistance approaches zero when the velocity approaches zero.

F. N. COLE,
Secretary.

THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION.

THE thirty-fourth regular meeting of the San Francisco Section was held at the University of California on Saturday, October 25. There were two sessions. The morning session was opened by the chairman of the Section, Professor Cajori, who later was relieved by Professor Blichfeldt.

The attendance was twenty-one, including the following fourteen members of the Society:

Professors R. E. Allardice, B. A. Bernstein, H. F. Blichfeldt, Thomas Buck, Florian Cajori, M. W. Haskell, L. M. Hoskins, D. N. Lehmer, W. A. Manning, H. C. Moreno, Dr. F. R. Morris, Professors C. A. Noble, T. M. Putnam, and Dr. Pauline Sperry.

The following officers were elected for the year: chairman, Professor H. F. Blichfeldt; secretary, Professor B. A. Bernstein; programme committee, Professors W. A. Manning, D. N. Lehmer, and B. A. Bernstein.

The dates of the next two meetings were fixed as April 10, 1920, and October 23, 1920.

The following papers were presented:

(1) Professor W. A. MANNING: "Doubly transitive groups with transitive subgroups of lower degree."

(2) Professor M. W. HASKELL: "On self-dual curves" (preliminary report).

(3) Professor W. A. MANNING: "Groups of degree n in which there is a circular permutation of n letters."

(4) Professor FLORIAN CAJORI: "Surveying and astronomical instruments used in America before the nineteenth century."

(5) Professor E. T. BELL: "On a certain inversion in the theory of numbers."

(6). Professor E. T. BELL: "On the enumeration of proper and improper representations in homogeneous forms."

(7) Professor E. T. BELL: "On proper and improper representations in certain quadratic forms of Liouville."

(8) Professor R. M. WINGER: "Some generalizations of the satellite theory."

In the absence of the authors the papers of Professors Bell and Winger were read by title. Abstracts of the papers follow below.

1. Professor Manning presented the following theorem:

Let a doubly (but not triply) transitive group of degree n have transitive subgroups $H, H_1, \dots, H_r$ of degrees $m, m + q_1, \dots, m + q_1 + \dots + q_r$, respectively, and of no other degree $< n$ and $> m$. Then H_{i-1} ($i = 1, 2, \dots, r$; $H_0 = H$) has at least

$$1 + (q_i + q_{i+1} + \dots + q_u) \left[\frac{1}{q_{u+1}} + \frac{1}{q_{u+2}} + \dots + \frac{1}{q_r} + 1 \right]$$

systems of imprimitivity of q_u letters which have one letter in common ($u = i, i + 1, \dots, r$). When $u = r$, no two of the above systems of q_r letters that have one letter in common have a second letter in common.

In particular this theorem asserts that H has $1 + q_1 + q_2 + \dots + q_r$ systems of q_r letters with one letter in common.

This problem was originated by C. Jordan. He showed (1871) that H_r , and a fortiori $H, H_1, \dots$, must have at least two systems of imprimitivity of q_r letters with one letter in common. Then B. Marggraff (1889) proved that H_r has $1 + q_r$ systems of imprimitivity of q_r letters with one letter in common. This result, as Marggraff did not fail to notice, reveals the presence in H_{i-1} of at least $1 + q_u/q_{u+1}$ systems of q_u letters with q_{u+1} letters in common ($u = i, i + 1, \dots, r - 1$). It was subsequently shown (Manning, 1906) that systems of imprimitivity of H_{i-1} of q_u letters each, with q_{u+1} letters in common, can be chosen in at least

$$1 + q_u \left[\frac{1}{q_{u+1}} + \frac{1}{q_{u+2}} + \dots + \frac{1}{q_r} + 1 \right]$$

ways ($u = i, i + 1, \dots, r$). The corresponding result, as stated in the present theorem, was suggested by a study of the group of isomorphisms of the elementary group of order 2^n .

2. Professor Haskell gave a preliminary account of the self-dual curves which are self-reciprocal with respect to a conic. He showed that any conic is self-reciprocal with respect to

each of a doubly infinite set of conics, and that this relation is mutual; that the self-dual cubic is self-reciprocal with respect to a singly infinite set of conics; and that the quintic with five cusps is self-reciprocal with respect to a single conic.

3. It was proved by Burnside that a simply transitive group of degree p (p a prime) is cyclic or contains an invariant subgroup of order p . He also proved that a simply transitive group of degree p^n in which there is a permutation of order p^n is necessarily imprimitive and compound. In his second paper Professor Manning announced the more general theorem:

If a simply transitive group of composite order contains a circular permutation of all its letters, it is compound.

4. Professor Cajori pointed out that the surveying and astronomical instruments used in America before 1800 were almost altogether from English makers—Heath, Short, Nairne, Bird, Sisson, Dollond, Ramsden. The chief American instrument makers were David Rittenhouse and Thomas Godfrey. Photographs of instruments were exhibited.

5. The inversion considered in Professor Bell's first paper is as follows. Define $f(x)$ to be regular if it exists and has a finite value when x is an integer > 0 , and $f(1) \neq 0$. Then there exists a unique regular f' , called the inverse of f , such that

$$\sum f(d)f'(n/d) = f(1) \text{ or } 0$$

according as $n = 1$ or $n > 1$, the Σ extending to all divisors d of the arbitrary integer $n > 0$. The f' thus defined is distinct from Berger's inverses. The theorem, which in some respects is similar to the well known inversion of H. F. Baker, is useful in many parts of arithmetic. The paper will appear in the *Tôhoku Mathematical Journal*.

6. Algebraic processes, elliptic function or other, do not easily yield the number of proper representations of an integer in a homogeneous form, but when applicable at all, give the total number of representations with great readiness. In the arithmetical treatment, it has been customary to deduce the total number of representations from the number of

proper representations. Inverting this procedure, Professor Bell, in his second paper, proves formulas of remarkable simplicity for the expression of the proper number in terms of the total, so that if the latter is within reach of analysis, so also now is the former. A few illustrations are given in the derivation of new results concerning 6, 8, 10 or 12 squares and other simple quadratic forms. The cases of 10, 12 squares present some unexpected singularities.

7. From 1860 to 1864 Liouville published in his *Journal* numerous theorems on the number of total and proper representations of integers in special quadratic forms of four and six indeterminates. His formulas for total numbers of representations were for the most part proved in 1890 by Pepin, those omitted being readily demonstrable by elliptic functions and other algebraic means. The formulas for proper representations have not hitherto been proved. Modifying the general principles of his second paper to fit Liouville's forms, Professor Bell demonstrates all of the unproved results very simply. The paper will appear in the *Journal de Mathématiques pures et appliquées*, (in French), and a fuller abstract shortly in the *Paris Comptes Rendus*.

8. Professor Winger's paper appeared in full in the November BULLETIN.

B. A. BERNSTEIN,
Secretary of the Section.

ON THE PROOF OF CAUCHY'S INTEGRAL FORMULA BY MEANS OF GREEN'S FORMULA.

BY MR. J. L. WALSH.

(Read before the American Mathematical Society December 30, 1919.)

It is well known that Cauchy's integral formula for an analytic function $f(z) = u(x, y) + iv(x, y)$ of the complex variable $z = x + iy$ is analogous to Green's formula for the functions u and v , and that moreover Cauchy's formula can be proved from Green's formula. Picard (*Traité d'Analyse* (1905), volume II, page 114) gives this proof *assuming that*

u and v have continuous second partial derivatives, which fact is proved later by means of Cauchy's formula. It is the purpose of the present note to give a modification of Picard's proof which shall not be open to this objection.*

Two functions $U(x, y)$ and $U'(x, y)$ are said to be conjugate in a region T of the x, y -plane if within that region they are continuous with their first partial derivatives and if in addition

$$(1) \quad \frac{\partial U}{\partial x} = \frac{\partial U'}{\partial y}, \quad \frac{\partial U}{\partial y} = -\frac{\partial U'}{\partial x}.$$

A function $V(x, y)$ is said to be harmonic in T if in T it is continuous together with its first and second partial derivatives and if it satisfies Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

Suppose U and U' are conjugate in a region T , and that V and V' are not only conjugate but also harmonic in T . Then if C is any contour bounding a region wholly in T , we shall have

$$(2) \quad \int_c \left[U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n} \right] ds = 0.$$

For the analysis given by Picard (l. c., page 10) establishes the following relations, where the double integrals are extended over the interior of C :

$$(3) \quad \begin{aligned} \iint \left[\frac{\partial U}{\partial x} \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \frac{\partial V}{\partial y} \right] dx dy &= - \int_c U \frac{\partial V}{\partial n} ds, \\ \iint \left[\frac{\partial U'}{\partial x} \frac{\partial V'}{\partial x} + \frac{\partial U'}{\partial y} \frac{\partial V'}{\partial y} \right] dx dy &= - \int_c U' \frac{\partial V'}{\partial n} ds. \end{aligned}$$

* Osgood, *Funktionentheorie* (1912), p. 679, gives such a proof due to Morera, but requiring the existence of a Green's function for the region. He also points out that Green's formula can be derived from that of Cauchy.

A proof for the circle, and hence for any region which can be conformally represented on the circle, is given by Kellogg, this *BULLETIN*, vol. 10 (1903-04), p. 255.

Compare also J. L. Walsh, "Note on Cauchy's integral formula," *Annals of Mathematics*, vol. 18, p. 79, where Cauchy's formula is proved by means of Cauchy's integral theorem and a proof of the mean value theorem for harmonic (or conjugate) functions due to Bôcher.

But by virtue of (1) and the corresponding relations between V and V' , the left-hand members of the two equations (3) are identical. Hence

$$\int_c U \frac{\partial V}{\partial n} ds = \int_c U' \frac{\partial V'}{\partial n} ds.$$

Also

$$\begin{aligned} 0 &= \int_c \frac{\partial(U'V)}{\partial s} ds = \int_c \left[U' \frac{\partial V}{\partial s} + V \frac{\partial U'}{\partial s} \right] ds \\ &= \int_c U' \frac{\partial V'}{\partial n} ds - \int_c V \frac{\partial U}{\partial n} ds. \end{aligned}$$

This immediately gives us equation (2).

Having proved this theorem,* we can continue by means of the analysis of Picard (l. c., page 15), setting $V = \log r$ and proving

$$U(a, b) = \frac{1}{2\pi} \int_c \left[\log r \frac{\partial U}{\partial n} - U \frac{\partial \log r}{\partial n} \right] ds.$$

Further developments (l. c., page 114) give us easily Cauchy's integral formula

$$f(z') = \frac{1}{2\pi i} \int_c \frac{f(z)}{z - z'} dz.$$

HARVARD UNIVERSITY.

* This was proved for a region bounded by a single contour, but there is no real difficulty involved in its later application to the multiply connected region.

A SET OF COMPLETELY INDEPENDENT POSTULATES FOR THE LINEAR ORDER η^* .

BY PROFESSOR M. G. GABA.

(Read before the American Mathematical Society September 4, 1919.)

PROFESSOR E. V. HUNTINGTON has published† three sets of completely independent postulates for serial order. His set A involves four postulates, which is as high a number of postulates as had been proved completely independent. In the present paper are given seven postulates which form a categorical and completely independent set for the linear order.

Our basis is a class of elements $[p]$ and an undefined dyadic relation (called 'less than') among the elements. If we are given two elements $p_1 p_2$ and if the relation p_1 less than p_2 holds, we will symbolize it by $p_1 < p_2$. If the relation p_1 less than p_2 does not hold, we will symbolize it by $p_1 \nless p_2$.

Our postulates are:

- I. If $p_1 < p_3$, then $p_2 \nless p_1$.
- II. If $p_1 \nless p_2$, then $p_2 < p_1$; p_1, p_2 distinct.
- III. If $p_1 < p_2$ and $p_2 < p_3$, then $p_1 < p_3$.
- IV. If $p_1 < p_2$, then there exists a p_3 such that $p_1 < p_3$ and $p_3 < p_2$.
- V. For every p_1 there exists a p_2 such that $p_2 < p_1$.
- VI. For every p_1 there exists a p_2 such that $p_1 < p_2$.
- VII. The class of elements $[p]$ form a denumerable set.

That the set is categorical follows from the fact that the seven postulates stated are the necessary and sufficient conditions for the linear order η . To show complete independence it will be necessary to cite 128 (2^7) examples showing all possible combinations ($\pm \pm \pm \pm \pm \pm \pm$) of our postulates holding and not holding. This is done by giving eight definitions of $<$, and sixteen sets of points such that each definition is applicable to every one of the sets, and every combination

* The linear order η is an ordered set equivalent to that of all the rational numbers.

† "Sets of completely independent postulates for serial order." This BULLETIN, March, 1917. This paper contains a bibliography of complete independence.

of definition of $<$ and set yields a different example. The eight definitions give the eight $(\pm\pm\pm)$ groups of cases for the implicational postulates I, II and III, whereas each of the sixteen sets gives all the eight cases where any particular set $(\pm\pm\pm\pm)$ of the existential postulates IV, V, VI and VII hold or do not hold.

For the independence examples, the set $[p]$ consists of points on a line such that

	IV	V	VI	VII	
1)	-	-	-	-	$p = -3, -2 \leq p \leq 2, p = 3$ and p real.
2)	-	-	-	+	$p = -3, -2 \leq p \leq 2, p = 3$ and p rational.
3)	-	-	+	-	$p = -3, -2 \leq p < 3,$ and p real.
4)	-	-	+	+	$p = -3, -2 \leq p < 3,$ and p rational.
5)	-	+	-	-	$-3 < p \leq 2, p = 3$ and p real.
6)	-	+	-	+	$-3 < p \leq 2, p = 3$ and p rational.
7)	-	+	+	-	$-3 < p \leq 1/2, 2 \leq p < 3,$ and p real.
8)	-	+	+	+	$-3 < p \leq 1/2, 2 \leq p < 3,$ and p rational.
9)	+	-	-	-	$-3 \leq p \leq 3,$ and p real.
10)	+	-	-	+	$-3 \leq p \leq 3,$ and p rational.
11)	+	-	+	-	$-3 \leq p < 3,$ and p real.
12)	+	-	+	+	$-3 \leq p < 3,$ and p rational.
13)	+	+	-	-	$-3 < p \leq 3,$ and p real.
14)	+	+	-	+	$-3 < p \leq 3,$ and p rational.
15)	+	+	+	-	$-3 < p < 3,$ and p real.
16)	+	+	+	+	$-3 < p < 3,$ and p rational.

A definition of $<$ requires that whenever we are given two numbers of our set $p_1 p_2$ we have a criterion whereby we can tell whether the relation $p_1 < p_2$ holds or does not hold. In all the eight definitions of $<$ the relation holds for any pair of numbers $p_1 p_2$ if it holds in the case of ordinary linear order,

	I	II	III	
1')	-	-	-	except $0 < 1, -1 < -2, 0 < -1$ and $0 < -2$.
2')	-	-	+	except $1 < -1, 1 < 0, 0 < -1, p_1 < -1, p_1 < 0, p_1 < 1, -1 < p_2, 0 < p_2$ and $1 < p_2; p_1 \neq -1, 0, 1; p_2 \neq -1, 0, 1$.
3')	-	+	-	except $0 < -m/2^n, n$ positive integer and m odd positive integer.
4')	-	+	+	except $p_1 < -1, p_1 < 0, p_1 < 1, -1 < p_2, 0 < p_2,$ and $1 < p_2; p_1 \neq 3; p_2 \neq -1, 0, 1$.
5')	+	-	-	and $p_2 - p_1 < 1/3$.
6')	+	-	+	and $p_2 - p_1 = m/2^n, n$ positive integer and m odd integer.
7')	+	+	-	except $0 < -m/2^n$ and $-m/2^n < 0, n$ positive integer and m odd positive integer.
8')	+	+	+	with no exceptions.

To illustrate: The independence example where postulates II, III, V, and VII hold and postulates I, IV and VI do not hold $(-++-+-+)$ is definition 4' used on set 6.

UNIVERSITY OF NEBRASKA.

CERTAIN PROPERTIES OF BINOMIAL COEFFICIENTS.

BY PROFESSOR W. D. CAIRNS.

(Read before the American Mathematical Society September 4, 1919.)

KENYON* has generalized results given by Chrystal† and others for the sum of products of equal powers of the terms of an arithmetic progression and the coefficients of $(x - y)^n$. In evaluating certain integrals connected with the probability curve,‡ the author has had to sum a similar set of products for the coefficients of $(x + y)^{2n}$; each binomial coefficient is multiplied by a given power of the number of that term from the middle term, e.g., the $(n + 1)$ th term by 0^q , the n th term by $(-1)^q$, etc., and the results added. One feature of this paper is the distinct gain from the symmetry thus utilized. A recursion formula is obtained for the sums and an expression for the term of highest degree in n , which is all that is needed in the desired application. For odd values of q only the terms from the middle to the end are used.

§1.

The general binomial coefficient of $(x + y)^{2n}$ is $\binom{2n}{n+k}$, where k is the number of the term counting from the middle term, and the sum to be evaluated for even values $q = 2p$ is

$$(1) \quad \sum_{k=-n}^n \binom{2n}{n+k} k^{2p}.$$

It is a familiar theorem that when $p = 0$ this sum is 2^{2n} . It proves that the method of differences can be used here provided that it be applied to the above series divided by 2^{2n} . Let

$$(2) \quad S(2n, q) \equiv \frac{1}{2^{2n}} \sum_{k=-n}^n \binom{2n}{n+k} k^q.$$

* Kenyon: *Proc. Indiana Academy of Science*, 1914.

† Chrystal: *Algebra*, Part II, p. 183.

‡ A paper read before the American Mathematical Society, September 5, 1918; this *BULLETIN*, December, 1919.

On replacing n by $n - 1$ the first difference is

$$\sum_{k=-n}^n \left[\frac{1}{2^{2n}} \binom{2n}{n+k} - \frac{1}{2^{2n-2}} \binom{2n-2}{n+k-1} \right] k^{2p},$$

(the first and last terms in the second part of the summation being zero); this reduces to

$$\sum_{k=-n}^n \frac{1}{2^{2n}} \binom{2n}{n+k} \left[1 - \frac{4(n+k)(n-k)}{2n(2n-1)} \right] k^{2p}$$

and hence to

$$\frac{2}{n(2n-1)} S(2n, 2p+2) - \frac{1}{2n-1} S(2n, 2p).$$

Writing the first difference as

$$D(2n, 2p) \equiv S(2n, 2p) - S(2n-2, 2p),$$

the recursion formula can be written in either of two forms

$$(3) \quad S(2n, 2p+2) = \frac{n}{2} S(2n, 2p) + \frac{n}{2} (2n-1) D(2n, 2p),$$

$$(4) \quad S(2n, 2p+2) = n^2 S(2n, 2p) - \frac{n}{2} (2n-1) S(2n-2, 2p),$$

where in particular by the elementary theorem referred to above

$$(5) \quad S(2n, 0) = 1.$$

The recursion formula and (5) give the results, after multiplying by 2^{2n} :

$$\begin{aligned} \sum_{k=-n}^n \binom{2n}{n+k} k^2 &= \frac{n}{2} 2^{2n}, \\ (6) \quad \sum_{k=-n}^n \binom{2n}{n+k} k^4 &= \frac{n(3n-1)}{4} 2^{2n}, \\ \sum_{k=-n}^n \binom{2n}{n+k} k^6 &= \frac{n(15n^2-15n+4)}{8} 2^{2n}, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

From these examples it may be inferred that the leading term in $S(2n, 2p)$ is $1 \cdot 3 \cdot 5 \cdots (2p-1) (n/2)^p$, and this can

be proved by using (3) or (4), or as follows. Since $S(2n - 2p, 2p)$ is obtained by replacing n in $S(2n, 2p)$ by $n - 1$, Mac-laurin's theorem gives

$$S(2n - 2, 2p) = S(2n, 2p) - S'(2n, 2p) + \frac{1}{2}S''(2n, 2p) - \dots$$

or

$$D(2n, 2p) = S'(2n, 2p) - \frac{1}{2!}S''(2n, 2p) + \dots,$$

the accents indicating differentiation with respect to n ; hence if the leading term of $S(2n, 2p)$ is assumed to be of degree p , that of $D(2n, 2p)$ will be of degree $p - 1$ and will be the leading term of $S'(2n, 2p)$. By (3) the leading term of $S(2n, 2p + 2)$ is

$$\begin{aligned} \frac{n}{2} \cdot 1 \cdot 3 \cdot 5 \cdots (2p - 1) \left(\frac{n}{2}\right)^p \\ + \frac{n}{2} \cdot 2n \cdot 1 \cdot 3 \cdot 5 \cdots (2p - 1) \cdot \frac{p}{2} \left(\frac{n}{2}\right)^{p-1} \end{aligned}$$

or $1 \cdot 3 \cdot 5 \cdots (2p + 1) (n/2)^{p+1}$; whence the theorem follows by mathematical induction. We have thus proved that the leading part of the sum (1) is $1 \cdot 3 \cdot 5 \cdots (2p - 1) (n/2)^p 2^{2n}$, the chief result of this section so far as concerns the properties of binomial coefficients.

With this result the evaluation of the series of integrals dealt with in a previous paper* is exhibited thus, employing the notation used at that time and observing the usual considerations of uniform convergence:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} x^{2p} dx &= \lim_{n \rightarrow \infty} \sum_{k=-n}^n \frac{\sqrt{\pi n}}{2^{2n}} \binom{2n}{n+k} (k\Delta x)^{2p} \Delta x \\ &= \sqrt{\pi} \lim_{n \rightarrow \infty} \sqrt{n} S(2n, 2p) \cdot \frac{\sigma^{2p+1}}{(n/2)^{(2p+1)/2}} \\ &= \sigma^{2p+1} \sqrt{2\pi} 1 \cdot 3 \cdot 5 \cdots (2p - 1). \end{aligned}$$

In the language of statistics one may from this say, for example, that the mean square deviation of the area under the curve $y = e^{-x^2/2\sigma^2}$ is σ^2 ; in other nomenclature we may say that the second, fourth, $\dots$ moments of this area are $\sigma^2, \sigma^4, \dots$. It is believed that the method of evaluating this already known integral is new.

* L. c.

By mathematical induction it is proved that the leading term in

$$\sum_{k=0}^n \binom{2n}{n+k} k^{2p+1}$$

is

$$\frac{1}{2} p! n^{p+1} \binom{2n}{n}.$$

Corresponding to the application made at the end of § 1, we have here

$$\begin{aligned} \int_0^\infty e^{-x^2/2\sigma^2} x^{2p+1} dx &= \lim_{n \rightarrow \infty} \frac{1}{\binom{2n}{n}} \sum_{k=0}^n \binom{2n}{n+k} (k\Delta x)^{2p+1} \Delta x \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} p! n^{p+1} \left(\frac{2\sigma^2}{n} \right)^{p+1} \\ &= \frac{1}{2} p! (2\sigma^2)^{p+1}. \end{aligned}$$

For example, since the area under the whole curve $y = e^{-x^2/2\sigma^2}$ is $\sigma\sqrt{2\pi}$, the "mean deviation" of this area is $\sigma\sqrt{2/\pi}$.

The products of the binomial coefficients by powers of terms of other arithmetical progressions do not seem to give simple results analogous to those obtained by Kenyon; this question is reserved for further study.

OBERLIN COLLEGE.

THE WORK OF POINCARÉ ON AUTOMORPHIC FUNCTIONS.

Oeuvres de Henri Poincaré, publiées sous les auspices du Ministère de l'Instruction publique par G. DARBOUX. Tome II, publié avec la collaboration de N. E. NÖRLUND et de ERNEST LEBON. Paris, Gauthier-Villars, 1916. lxxi + 632 pp.

THE collected works of Poincaré will fill some 10 volumes, of which the one before us is the first to be published. It contains the principal papers written by him in the field of

automorphic functions, where some of his most brilliant early work lies. According to the statement of Darboux in the preface, this Volume II appears first with the hope that it may stimulate mathematicians to active work in that field.

An invaluable and necessary revision with critical notes is supplied by Nörlund.

The many remarkable eulogies pronounced shortly after the death of Poincaré testify to a very wide recognition of his dominating position in the mathematical world. In one of the best of these, Volterra says: "if we were to characterize the recent period of the history of mathematics by a single name we should all give that of Poincaré."* Among these appreciations that of Hadamard† may be mentioned here for its critical value, while the admirable "Eloge Historique" of Darboux deserves its introductory place in the volume.

It is impossible to give any satisfactory idea of the achievements of Poincaré in the space of a single essay. The appearance of his collected papers arranged by subjects will furnish an occasion for a more leisurely and critical review of his work in its relation to the mathematics of the time. It is my purpose to attempt such a review.

An immediate stimulus for Poincaré's researches in the domain of automorphic functions was the question proposed for the Grand Prix of 1880 in the mathematical sciences: to complete in some important point the theory of ordinary linear differential equations. Partly on account of incompleteness in his development of the new functions, the prize was not awarded to Poincaré but to G. H. Halphen. An extract from this initial attempt of Poincaré is to appear in volume 39 of the *Acta Mathematica*.

In order to understand the nature of the advance made by Poincaré it is necessary to go back to the nearly contemporary work of Schwarz, Fuchs, Schottky, and Klein.

It was obvious after Riemann's time that the inverse of the ratio of a pair of solutions of an ordinary linear differential equation of the second order was an automorphic function, i.e., one unaltered by a group of linear fractional transformations. Examples of such functions were at hand

* Volterra, "Henri Poincaré." A lecture delivered at the inauguration of Rice Institute. Translated by G. C. Evans. Rice Institute Pamphlets, Vol. I (1915), No. 2, pp. 133-162.

† "Henri Poincaré: le mathématicien," *Revue de Métaphysique et de Morale*, vol. 21 (1913), pp. 617-658.

in the trigonometric and elliptic functions, the elliptic modular functions, the more general triangle functions first classified by Schwarz, other functions due to Schottky, and finally the rational automorphic functions whose beautiful algebraic and geometric relations were developed by Klein. When considered in relation to the illuminating work of Fuchs on ordinary linear differential equations, these examples made it an obvious probability that an extensive theory of single-valued transcendental automorphic functions awaited development. Fuchs pointed out this field explicitly first in 1880, but Poincaré's account of the genesis of his ideas indicates his essential independence of this paper.*

The general transcendental automorphic functions, however, were not immediately approached, because of a certain gap to which we will now refer.

If we take the singular points of the differential equation and the characteristic exponents as real, it is an obvious deduction from the work of Fuchs before 1880 that the inverse function maps a circular polygon conformally upon a Riemann's surface, and that by the process of analytic extension other polygons which are obtained by linear fractional transformations, arise. An immediate necessary condition that there is no overlapping of the polygons is that the angles at the vertices of the first polygon are either 0 or fractional parts of 2π . The precise further conditions for non-overlapping were known only in the case of the rational automorphic functions, treated by Klein, where the polygons were representable as spherical polygons so that the ordinary formulas of spherical trigonometry were available. But the earlier work of Fuchs made it clear that single-valued automorphic functions exist always when there is no overlapping.

Thus the moment at which Poincaré reached scientific maturity was a most favorable one in which to create a general theory of these functions which were challenging the attention of mathematicians.

Here it seems worth while to call attention to a situation which often exists in mathematics but scarcely ever exists in any other field of science. It is sometimes the case that several mathematicians are in possession of much new material save for a single link of reasoning. However, on

* "Science et Méthode," Flammarion, Paris, 1908, Chap. 3; or pp. lvi-lviii of Darboux's "Eloge Historique."

account of self-imposed requirements of completeness and rigor, no claim is made for credit.

It was precisely such a state of affairs that existed in the domain of the transcendental automorphic functions when Poincaré began his work. The algebraic relation between automorphic functions with the same group, the uniformizing properties of these functions, and their use in solving linear differential equations are illustrations of material of this sort.

It was by his use of non-euclidean geometry that he overcame the one fundamental difficulty referred to above. How then did these geometric ideas enter?

The region of the complex plane under consideration is the interior of a fixed circle and the bounding circles of the polygons referred to are all orthogonal to it. Furthermore, the linear fractional transformations of these polygons are such as to leave the fixed circle invariant. Here, then, are the abstract features of the non-euclidean geometry of Lobachevski, namely, a three-parameter group of transformations which may be interpreted as the group of rigid motions if every circle orthogonal to the fixed circle be regarded as a straight line, and if angles be interpreted as usual.

In recognizing this fact Poincaré faced an extensive vista of possible developments in the field of automorphic functions, and incidentally obtained a simple new geometric representation of the hyperbolic plane. It was natural to connect this geometric view with the highly suggestive work of Klein. Thus we are not surprised to find Poincaré a master of the powerful weapons furnished by projective geometry at the outset.

From this new point of view the difficulty of overlapping presents an absolutely concrete aspect. The polygon appears as an ordinary polygon in the non-euclidean plane, and the problem is to determine when the entire plane can be filled by a network of congruent polygons. The precise analytical conditions are seen to be algebraic.

Although the essential kernel of the advance of Poincaré was only a single step, he must be regarded as the true founder of the theory of the general transcendental automorphic function. A somewhat analogous situation is presented by the founding of the calculus. Without doubt many mathematicians were in possession of most of the essential ideas of

the calculus, but failed to discern the possibility of elevating it to an independent discipline by the invention of a suitable symbolism. Nevertheless there is general agreement that Newton and Leibniz were the originators of the calculus, precisely because they did see this possibility.

Had Newton merely developed the simpler parts of the calculus of fluxions, his title to preeminence would have lost some of its validity. But Newton not only conceived the possibility but carried it to fulfilment by developing an adequate technique and applying it to the problems of celestial mechanics. Likewise, Poincaré merits great credit because, once in possession of the fundamental new weapon of attack, he proceeded to develop the theory of the transcendental automorphic functions to a remarkable degree.

In the further development of his new ideas Poincaré faced two possibilities. On the one hand, following the general course of Riemann, Schwarz, Fuchs, Schottky, and Klein, he might develop a general theory based on existence theorems for conformal mapping. This was a possible course as he saw plainly. Thus he declares in an early paper (*Mathematische Annalen*, 1882, translation) "From the existence of discontinuous groups one might without doubt, by processes analogous to those of Schwarz, deduce that of single-valued functions reproduced by the substitutions of the group; but one would have then no explicit expression for these transcendental functions. Moreover it is better to use other methods." The alternative method was based upon certain explicit series invented by Poincaré before he noted the geometric relation outlined above, and to these series we now turn.

In the case of an ordinary finite group of linear fractional transformations the sum of a rational function $H(z)$ of z , taken for any value of z and its transformed arguments z_k under the group, clearly forms an invariant function. Similar infinite series exist in the case of the series for the elliptic functions. These infinite series, invented by Cayley and Eisenstein, diverge in certain cases although their derivative series converge. On the basis of such series Weierstrass constructed his theory of the elliptic functions.

Suppose that in an analogous way we form the series formally invariant under an arbitrary infinite discontinuous group of linear fractional transformations leaving invariant a circle. The image points z_k of z will approach the invariant circle,

as k becomes infinite, and thus the k th term $H(z_k)$ of the series will remain finite, at least if the rational function H with which we start has no poles upon that circle. But there is no reason to assume that this series converges.

If, however, the series be formally differentiated term by term, the typical terms of the new series are the product of the derivative rational function $H'(z_k)$ taken at the image point z_k and the derivative dz_k/dz . The modulus of the second factor represents the ratio of arc lengths of corresponding similar infinitesimal figures. The area integral of the squared modulus of this factor over any part C of a fundamental region yields the ordinary area of the image C_k . But the sum of all the areas C_k is less than the area of the invariant circle. Hence the series

$$\Sigma \iint \text{mod } (dz_k/dz)^2 dx dy$$

converges. An easy extension of this argument shows that $\Sigma \text{mod}(dz_k/dz)^2$ also converges. Thus the factor dz_k/dz approaches 0 as k becomes infinite.

Now if such a derivative series converges it does not represent an automorphic function but a function I with the property

$$I(z_k) = I(z)(c_k z + d_k)^{-2},$$

where

$$z_k = (a_k z + b_k)/(c_k z + d_k)$$

with $a_k d_k - b_k c_k = 1$. Thus one is led to consider series whose typical term is of the form $H(z_k) (dz_k/dz)^m$ which will converge if $m \geq 2$, and which are multiplied by $(c_k z + d_k)^{-2m}$ when z is changed to z_k .

Moreover the functions defined by these theta series of Poincaré play the rôle of the elliptic theta functions for elliptic functions and are suited to form a basis for the general development of the theory of automorphic functions; in fact the quotient of two such functions having the same value of m is an automorphic function. No other completely explicit expressions for the automorphic functions have as yet been found.

All of these facts can be looked at from the point of view of homogeneous variables and then take on a more elegant form, but it seems probable that the development of Poincaré's own ideas concerning these series was approximately along the line of least resistance just outlined.

With these two touchstones of the theory in his possession—namely, the interpretations of non-euclidean and projective geometry, and the explicit series above mentioned—the analogy of the theory of automorphic functions and the elliptic functions became doubly apparent. The interrelation of elliptic functions with algebraic functions and their integrals, with linear differential equations, and with number theory were seen by him to admit of complete generalization, and the range of knowledge called for in order to carry out this development was just that which Poincaré had come naturally into possession of at Paris.

His five long papers in the early volumes of the *Acta Mathematica* give a first approximation to the theory of the automorphic functions even at its present degree of development. A large variety of important complementary results have been added, mainly by German mathematicians, who have used homogeneous variables, and have developed an independent theory from the point of view of Riemann alluded to above.

In the first of these extensive papers entitled “Théorie des groupes fuchsien,” Poincaré developed explicitly the forms of polygons and networks of congruent polygons in the non-euclidean plane from his geometric point of view. He called discontinuous groups which possess an invariant circle Fuchsian groups in honor of Fuchs, and the corresponding automorphic functions Fuchsian functions. Perhaps it is better to use the terminology of Klein, and speak of groups with principal circle and automorphic functions with principal circle. Poincaré’s classification of the types of groups is suggestive but has not the definitive form which has been given it by Fricke.

In a second paper “Sur les fonctions fuchsien” he considers the series above obtained and develops the main facts concerning the functions which they define. The chief difficulty is caused by the fact that such a function may vanish identically. By means of the theory of these functions a theory of the automorphic functions can also be derived. A cardinal fact to prove here is that every automorphic function can be represented as the quotient of two such series. Conversely, one can pass from the automorphic function back to the theta series of Poincaré by a process of differentiation.

A first fundamental application of these functions, noted in the same article, is to the uniformization of algebraic functions. Between any pair of automorphic functions with the same group an algebraic relation obviously exists, just as an algebraic relation exists between two doubly periodic functions with the same period parallelogram. The automorphic functions uniformize this algebraic relation in the sense that each variable is expressible by means of an automorphic function belonging to the same group. In this way the Riemann surface is mapped conformally upon the circular polygon referred to earlier. Thus the important question arises at once: is it not possible to uniformize every algebraic relation by means of such function? An early count of constants showed Poincaré that this was in all probability the case. A second application of importance lay in the integration of the corresponding linear differential equation of the second order.

In a third paper "*Mémoire sur les groupes kleinéens*" Poincaré treated the more general automorphic group in which there was no invariant circle, when the same theta series were available. These groups and the corresponding automorphic functions were called Kleinian by him, but are perhaps better designated as groups and automorphic functions without principal circle. The fundamental geometric results which made an attack on these more general functions possible were due to Cayley and Klein. Spatial non-euclidean geometry enters, inasmuch as the totality of linear fractional transformations of the complex plane is isomorphic with those projective transformations of a quadric in space which leave the quadric invariant.

In two further papers, "*Sur les groupes des équations linéaires*" and "*Mémoire sur les fonctions zétafuchsiennes*," Poincaré attacks the problem of uniformizing algebraic functions and the integrals of linear differential equations with algebraic coefficients. It is easy to see that if the function has only three branch points or if the linear differential equation has rational coefficients and three singular points, regular or irregular, then such a uniformization is possible by means of the elliptic modular function, provided the images of the three singular points lie at the vertices of the fundamental triangle. This admits of obvious generalizations. In his attempt to establish such generalizations he proves first that

the characteristic constants of the monodromic group are entire functions of the literal coefficients, and then employs the "method of continuity" of Klein. The satisfactory application of this method has proved difficult. Poincaré gave explicit series analogous to his theta series in terms of which such uniformization for the differential equation is possible.

In attempting to appraise the importance of the new functions discovered by Poincaré, one must always remember that these functions form a natural extension of many of the functions treated earlier in analysis. Moreover, they afford a very simple systematic method of approach to the theory of the algebraic functions and their integrals, and add an illuminating chapter in the theory of ordinary linear differential equations with algebraic coefficients. But the process of natural generalization appears to terminate with the invention and investigation of these functions, just as the theory of the elliptic functions reached its conclusion earlier. I do not expect to see large further developments take place, notwithstanding the hope to the contrary implied by Darboux in the preface.

The mathematician of the future, having before him such completed theories, will seek on the one hand to clarify and simplify, and on the other hand to use only as much of them as is really vitally related to further advance.*

If this be true it may be predicted that the elliptic modular functions and the triangle functions will maintain an important position on account of their intrinsic importance and as the best examples of transcendental automorphic functions with principal circle, while the more general functions discovered by Poincaré will receive only such consideration as is necessary to establish their position in the hierarchy of analytic functions.

GEORGE D. BIRKHOFF.

* A treatment of automorphic functions in this spirit is given in the concluding chapter of Professor Osgood's "Lehrbuch der Funktionentheorie," vol. 1 (second edition), Teubner, Leipzig, 1912. See also his paper "On the uniformization of algebraic functions," *Annals of Mathematics*, ser. 2, vol. 14 (1912-1913), pp. 143-162.

A BRIEF ACCOUNT OF THE LIFE AND WORK OF THE LATE PROFESSOR ULISSE DINI.

BY PROFESSOR WALTER B. FORD.

(Read before the American Mathematical Society September 3, 1919.)

THE death of Professor Ulisse Dini in his native city of Pisa, Italy, on the 28th of last October marked the passing of one whose name has long been familiar to the mathematical fraternity of the entire world and it therefore seems fitting, despite the wide separation geographically between the scene of his labors and our own, that a brief account of his life and work be given at this time in America and a small measure at least of tribute be rendered to his genius.

Dini was born on the 14th of November, 1845, of parents highly respected but of very moderate circumstances. He early manifested an unusual activity of both mind and body, thus commanding the attention and admiration of his instructors and masters who foresaw for him a future of extraordinary promise. Upon entering the neighboring university of Pisa his marked capabilities in mathematics were soon recognized and he shortly became the favorite pupil of Professor Betti, well known as one of the leaders in mathematical instruction and research at this period in Italy. Under such auspices and at the uncommonly early age of nineteen years, Dini attained the laureate and received directly afterward a government scholarship enabling him to continue his studies for a year at Paris. During this brief foreign sojourn he came chiefly under the influences of Bertrand and Hermite and formed an intimate friendship with them which extended long into later years. Returning forthwith to Pisa, his active career as teacher and investigator began in the year 1866 while he was yet scarcely of age and continued almost without interruption for over fifty years, being characterized throughout by the utmost zeal and devotion, not only to mathematical science, but to his native country, city and university, from all of which he received in turn the highest honors they could bestow.

Such in brief was Dini's career. If we inquire in a more detailed sense what his relations to mathematics actually

were and what his special achievements, it may be noted in the first place that his earliest researches lay in the field of infinitesimal geometry; more specifically, in the determination of the form and properties of certain partial differential equations which arise in the theory of applicable surfaces. His work in this connection, though extending over no more than six years (1864-1870), gave rise to some eighteen memoirs dealing chiefly with general problems in the theory of curvature and geodesics, some of which had been proposed earlier by Beltrami. At this early period of his life, however, Dini had not yet begun the researches for which he is to be regarded as famous nor had he in fact even entered seriously into that broad field, namely pure analysis, in whose development he was destined soon to play an active part. The transition of his interest and labors to this latter field took place about 1870. At this comparatively early date it will be recalled that the newer and more rigorous analysis (the so-called "modern analysis" of to-day) was but little known to the world at large, the spirit of its methods being virtually confined to the limited school of pupils immediately surrounding Weierstrass in Germany. Nevertheless, once Dini had turned his efforts in this direction, he appears to have reached within a remarkably short time a full appreciation of these newer ideas and methods. In fact, he straightway acquired such a critical insight into their significance and developed such ability and confidence in their use that he was soon independently at work carrying out for himself their manifold consequences, especially their bearing upon those concepts which lie at the foundation of analysis. Thus, by the year 1877, or seven years from the time he began, he published the treatise, since famous, entitled *Foundations for the Theory of Functions of Real Variables* (*Fondamenti per la teoria delle funzioni di variabili reali*). Much of what Dini here sets forth concerning such topics as continuous and discontinuous functions, the derivative and the conditions for its existence, series, definite integrals, the properties of the incremental ratio, etc., was entirely original with himself and has since come to be regarded everywhere as basal in the real variable theory. The book has, in fact, served as a model the world over and even at this date, which is more than forty years since its publication, it still affords one of the best available expositions of the basal concepts of analysis

as regarded from the standpoint of modern rigor, evidence of which may be found, for example, in the fact that as late as 1902 an authorized translation of the work was published in German.

The fact that Dini, though situated at the time far from any natural sources of information, was able to arrive so early at a thoroughgoing appreciation of the newer methods in analysis and to produce straightway so unique and valuable a work as the "Foundations" leads one naturally to inquire somewhat farther than we have thus far indicated into the attending circumstances. One may well ask, for example, what particular problem or incident served as his starting point and thus became his first inspiration. Dini's own explanation of this, as given in after years to inquiring friends, was somewhat as follows: While still a pupil at the university he came to feel serious doubts as to the actual meaning of some of the most fundamental concepts commonly employed in the calculus. They did not seem to be capable of the sort of definite, working formulation which one has a right to expect of every concept employed in mathematical science. While thus concerned in his own mind, he learned some years later from certain memoirs of Hankel, Schwartz, and Heine that similar doubts had been experienced by others and had already been resolved, at least in part, by Weierstrass and his school in Germany. Thereupon he obtained from this distant source such further information as was then available and, though meager and confined to special problems only, it was sufficient to impart the spirit of the newer methods and to convince him of their far reaching possibilities and consequences. He then began, single handed and alone, that remarkable series of studies and researches which culminated but a few years later in his treatise above mentioned on the Foundations.

With the assurance once gained that he was working upon well-grounded principles and definitions, Dini next proceeded to apply them in an extended and detailed sense. Thus, during 1877-78 he reworked and treated in his lectures a wide variety of topics taken from the usual course in higher analysis. In particular, he here gave for the first time a rigorous treatment of the general theory of implicit functions. His numerous researches at this period were left unpublished, however, being preserved only in lithograph form. Not

until the later years of his life did he undertake the considerable task of arranging the whole for publication, but it may now be found, together with much supplementary material, in his four large volumes published as late as 1915 entitled *Lessons in Infinitesimal Analysis* (*Lezioni di analisi infinitesimale*).

Following the studies of a very general nature above referred to, Dini turned his attention in 1878, or thereabouts, to a more specific field; namely, to a critical investigation of the convergence and other important properties of Fourier series. The result appeared two years later in the form of another treatise, this being entitled *Fourier Series and other Analytical Representations of the Functions of a Real Variable* (*Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale*). Here we meet with an entire book of 328 pages representing a single piece of sustained, original research and embracing results of a remarkably high order. Not only is the convergence of Fourier series studied from many points of view, but a complete and satisfactory discussion is here given for the first time of the convergence of various other allied developments well known in mathematical physics, such as the developments of an arbitrary function in terms of Bessel and Legendre functions. Moreover, a general method is finally set up for the examination of all series of the Fourier type (series in terms of normal functions). Dini's wonderful powers, both critical and inventive, doubtless reach their highest realization in this work, though, unlike the *Foundations*, the style of presentation is not well calculated to bring out the merit of the whole. This feature is largely due, as noted in the preface, to the fact that the publication proceeded at intervals in such a way that no opportunity was afforded to put the earlier parts of the work in proper adjustment with the later ones.*

* As the development of an arbitrary function in terms of normal functions has been extensively considered in recent years, especially from the standpoint of the modern theory of integral equations, a word should perhaps be said as to the relation between these more recent studies and those of Dini above mentioned. Taking the question of convergence alone, the results of the more recent studies may be said to reach their culmination in the researches of Haar, *Math. Ann.*, vol. 69 (1910), p. 331 and vol. 71 (1911), p. 38. These, while exceedingly general, do not, however, include such important special cases as the Bessel and Legendre developments mentioned above. For a summary statement of Haar's results see Bôcher's address at the international congress of mathematicians at Cambridge (England), 1912, pp. 29-30, and for the underlying reason to account for the limitations which such results present, see the same address, p. 1,

Thus far we have mentioned only Dini's larger treatises. Space would hardly permit of a detailed consideration of his shorter memoirs, and fortunately this is not necessary, since the list, which comprises in all about sixty papers, has recently been prepared by his colleague, Professor Luigi Bianchi, and published along with a more extended account of his entire scientific achievements in the proceedings of the Accademia dei Lincei of last February. Suffice it to say that these memoirs pertain largely to infinitesimal geometry, to the theory of functions of a complex variable and to the study of differential equations, total and partial.

We might close our brief account at this point were it not for the fact that an adequate conception of Dini's activities can scarcely be gained from an examination of his scientific career alone. Dating from about the year 1880, he was chosen time and again to occupy positions of honor and trust in the affairs of his city, province and nation. In particular, he was at the time of his death vice-president of the national council of public instruction, a senator of the kingdom and the director of the normal college (Scuola Normale Superiore) of Pisa, in addition to his position as professor at the university. His death, therefore, was deeply mourned by a wide circle of people and institutions all of which have since united in an effort to erect a permanent monument within the city of Pisa that shall symbolize not only his genius in science but the love and esteem in which he was universally held by his countrymen.

UNIVERSITY OF MICHIGAN.

SHORTER NOTICES.

Commercial Algebra. By WENTWORTH, SMITH and SCHLAUCH.
Boston, Ginn and Company, 1917-1918.

THIS work appears in two volumes, Book I designed for the first year course of a commercial high school, and Book II a text for advanced classes in the same lines.

statement (4) at the bottom of the page. The excluded cases of importance have usually been considered (though not by Dini) only through special methods adapted to the case in hand. Thus, see Hobson, *Proc. London Math. Soc.*, vol. 7 (1908), pp. 359-388; also C. N. Moore in various articles in the *Trans. American Math. Soc.* dating from 1907.

As compared with the usual first book in algebra, there are a few departures in Book I. The authors call attention in the preface to the omission of "a large amount of non-essential work in factoring, fractions, and complicated equations." The problems, too, are chosen with a view to familiarizing the student with commercial phraseology and presenting appropriate applications to such topics as shop formulas, daily balances, cost-rate factors, commission formulas, etc. One of the newer features is the application of simple equations to problems in standard dietaries and ratio of solvency. Graphic work is introduced in elementary form in Chapter III, and considerable practice is given in pictograms and cartograms, with special reference to advertising and statistics. In Chapter XIII a more extended treatment of graphic methods is given, including the topics of wage graphs, interpolation, net profit, and interest.

In Book II the first four chapters present the topics of algebra necessary as a preparation for the applied work, namely, powers and roots, logarithms, slide rule, and series. The remainder of the book treats both theoretically and practically the subjects of compound interest, equation of payments, annuities, amortization, depreciation, bonds, life insurance, and alignment charts. This volume contains tables of compound interest, present value, annuities, and logarithms.

While these volumes follow rather closely the lines of texts already on the market, both in content and method of presentation, a particular point has been made of so simplifying the work as to adapt it to the needs of the high school. This adaptation appears in the omission of the more advanced work in series, the choice of problems, and the occasional replacing of abstract notation by significant letters, as in the formulas connecting nominal and effective rates of interest.

The two books are a useful contribution to the organization of high-school courses, in that they meet present-day demands for practical topics, without sacrificing an intelligent understanding of the tools used—an end which can be gained only by a careful development of the theoretical side.

The publication of such texts as this may be taken as a hopeful sign—not that the curriculum is becoming over commercialized, but that the commercial world is finding it

increasingly necessary that its people shall be trained in logical mathematical thought.

F. E. ALLEN.

The Training of Teachers of Mathematics for the Secondary Schools of Countries represented in the International Commission on the Teaching of Mathematics, by RAYMOND CLARE ARCHIBALD. With the Editorial Cooperation of D. E. SMITH, W. F. OSGOOD, J. W. A. YOUNG, members of the Commission from the United States. Washington, Government Printing Office, 1918.

THIS well-executed piece of work includes an able discussion of the training of teachers in Australia, Austria, Belgium, Denmark, England, Finland, France, Germany, Hungary, Italy, Japan, the Netherlands, Roumania, Russia, Spain, Sweden, Switzerland, and the United States, as well as appendices with typical examinations as given in England, France, Germany, and Japan. The Summary is also a valuable feature.

No one can deny the assertion that most of the countries discussed have far higher standards for their teachers of mathematics in secondary schools than we have in the United States. In this connection the interesting fact is brought out that there were in the United States, in 1913-1914, 13,714 "accredited secondary schools." Similar statistics for other countries, unfortunately not given, would have been of great significance.

Our need for well-trained teachers is not less than that of any other country. However, our problem to obtain and train these teachers has been fundamentally different from the European problem of obtaining teachers. We have been trying to educate all the children of all the people, whereas no great European country has made any similar attempt. England, France, Germany, Italy and Russia are all on the same footing, in training only a select few and differentiating upon entrance into the most elementary school the elect, those who may go on, from those who will not go on into secondary schools. With this system, the relatively few teachers needed can be given a higher and more intensive training.

The effect of the war upon the educational systems of European countries will be great. Undoubtedly the present industrial unrest will culminate in the democratization of the

schools of all the world. More than ever it is incumbent upon American leaders in education to exert every effort to improve the preparation of the teachers and the status of the profession, for it is to our system of education that Europe will turn for guidance.

The recent tremendous increases in enrollment of students in our universities includes fortunately large numbers who are taking up the study of mathematics. It is somewhat significant that this increase is affecting only the universities, not the normal schools. Undoubtedly, the day is not distant when our universities and colleges will be called to train not only high school teachers, but also the "junior high school" teachers. The nature of this training for the instructors in mathematics in the secondary schools needs systematic and continued study, taking into account the changes in the world in which we live.

This study under review, while reflecting a world which has passed, will nevertheless continue for many years to be of great value to all concerned with the preparation of our teachers of mathematics.

L. C. KARPINSKI.

Theory of Maxima and Minima. By HARRIS HANCOCK.
New York, Ginn and Company, 1917. v + 193 pages.
Price \$2.50.

THE student of mathematics meets the interesting subject of maxima and minima early in his first course in calculus. The subject appears again and again in his courses in advanced calculus and functions of a real variable, each time carrying the student a little deeper into the theory, but rarely giving him opportunity to view the subject as a whole. The English reading student finds it particularly difficult to study the theory for functions of two or more variables as expounded by Scheefer, Stolz, von Dantscher, and Weierstrass. Professor Hancock's book is for this English reading student.

The opening chapter discusses functions of one variable, first taking up functions having complete derivatives throughout the interval in question (ordinary maxima and minima). This covers the usual discussion in a first course in calculus with a somewhat more mature reader in view. Then follows a discussion for functions having derivatives only for definite values of the variable, or having one-sided derivatives (ex-

traordinary maxima and minima). A statement of the theorem concerning the existence of the upper and lower limits of a continuous function in a definite interval is included without proof.

Chapter II is a preliminary chapter on functions of several variables, outlining in a broad way the general problem. Chapter III is limited to functions of two variables, treating in a general way the different cases that may arise, pointing out the difficulties of the ambiguous case, discussing the errors that are in the literature and noting the various attempts to improve the theory. This leads up to Chapter IV in which Scheefer's theory for two variables is developed in detail with a general outline of Stolz's extension to the case of three variables. Von Dantscher's theory is included for comparison purposes. These three chapters are the most important in the book. They give in sufficient detail the theorems of Scheefer, Stolz and von Dantscher together with a discussion of the errors made by earlier writers and carried on into many of the modern textbooks on the calculus. In particular the so-called ambiguous case, around which most of the difficulties center, is treated in a fuller manner than in any other work in English that the reviewer has examined. On page 34, the author quotes from Professor Pierpont's article in Volume IV of the BULLETIN, "Our English and American authors seem to be ignorant of these facts." At first the reviewer was inclined to look upon this quotation written in 1898 as rather historical, but a study of the English and American books available failed to bring forth a full discussion of the maxima and minima of functions of two variables. However in many cases the reason was not that the authors seemed "to be ignorant of these facts," but simply that they lacked the space for a full treatment. This full treatment is here.

The last four chapters in the book follow very closely the work published by Professor Hancock in 1903, *Lectures on the Theory of Maxima and Minima of Functions of Several Variables* (Weierstrass' Theory).

These chapters, which follow the lectures of Weierstrass, treat only the ordinary cases where the functions are everywhere regular and where the forms are either definite or indefinite, excluding the semidefinite form characterizing the ambiguous case. Some interesting additions have been made

to the older work. Among them are two interesting light theory problems prepared by Professor Brand. One problem is concerned with the necessary and sufficient conditions for minimizing the length of a ray of light from one point to another after reflection at a point on the surface $F(x, y, z) = 0$. The other is a refraction problem in which the time between two points is to be a minimum. Hadamard's determinant problem is also included.

Throughout the book are many examples and problems illustrating points in the theory, some worked out in detail.

A. R. CRATHORNE.

From Nebula to Nebula, 4th Edition. By GEORGE HENRY LEPPER. Pittsburgh, Pa., privately printed, 1919. 401 pages.

(FROM the introduction) "The object of this work is two-fold: (1) to present a revaluation of the time-honored doctrines upon which modern theoretical astronomy is based, and (2) having shown wherein they are defective, to propose a new and far more comprehensive system, revealing the entire visible universe in the philosophic aspect of a single unit coordinated throughout, as *a priori it must be*, by a single dynamical force."

It is not possible, within the limits of available space, to give a complete analysis of this book. A few quotations will indicate some of the author's conclusions, but the proofs which he gives must be omitted. These quotations are so chosen that the lack of context need do no injustice to the author.

"(The earth . . .) is generously endowed with liquid oceans, which she is compelled to shift constantly from the sunlit side in her efforts to preserve her center of gravity at the lowest point. It is this struggle for equilibrium . . . which I assign as the major cause of the earth's diurnal rotation."

"I contend, with Aristotle, that rest is the natural state of matter."

"I have been enabled . . . to demonstrate the sun's path to be that of an immense spiral with a diameter of 1,530,000 million miles and a total coil length of nearly seven trillions."

"Paradoxical as it may sound, . . . I contend that the sun and not the moon is the *vera causa* of the tides."

After attempting to prove that tides cause the rotation of

the earth the author states: “. . . it becomes quite evident that the earth is not turning on its axis simply ‘because it can not stop a motion that never was started,’ but because Nature intelligently supplies a quarter quadrillion horse power engine to do the work.”

This review may well end with the following quotation which Lepper himself gives, ascribing it to the director of the Smithsonian Institute, and which is illustrated by this volume. “Every large scientific institution or observatory has almost daily communications from persons of very moderate attainments who presume to question, nay rather to spurn, the most well-attested facts of human knowledge. . . . No argument can refute them because they have not the requisite learning to comprehend it, which is no disgrace but which should make men *modest* enough to have faith in *those who excel them immeasurably*. . . .”

F. H. SAFFORD

A Treatise on the Analytical Dynamics of Particles and Rigid Bodies; with an Introduction to the Problem of Three Bodies.
By E. T. WHITTAKER. Second edition. Cambridge, University Press, 1917. xii + 432 pp.

THE second edition differs from its predecessor mainly in that references to more recent researches are given together with a brief outline in certain cases.

The book is invaluable as a condensed and suggestive presentation of the formal side of analytical dynamics. There is serious need for a complementary type of treatise in which the main emphasis is laid upon the deeper qualitative side which has played an increasingly larger part since the work of Poincaré.

As a single instance of the incompleteness of Whittaker's treatment in this respect, the fact may be mentioned that in his final chapter he treats the trigonometric series precisely in the spirit of Delaunay and does not even mention that these series are generally divergent nor refer to their asymptotic properties.

G. D. BIRKHOFF.

NOTES.

THE concluding (October) number of volume 20 of the *Transactions of the American Mathematical Society* contains the following papers: "Line-geometric representations for functions of a complex variable," by E. J. WILCZYNSKI; "On the combination of non-loxodromic substitutions," by E. B. VAN VLECK; "On a general class of integrals of the form $\int_0^\infty \varphi(t) g(x+t) dt$," by R. D. CARMICHAEL; "Transformations of surfaces applicable to a quadric," by L. P. EISENHART; "Note on two three-dimensional manifolds with the same group," by J. W. ALEXANDER; "A general system of linear equations," by A. J. PELL; Indexes, volumes 11-20.

AT the Brussels meeting of the National Research Council several International Unions were formed. For other unions, where the representation was insufficient divisional committees were formed. In mathematics a divisional committee was nominated and charged with the duty of circulating a draft of statutes so as to obtain the opinion of the principal bodies concerned. M. DE LA VALLÉE POUSSIN was elected president, and the following secretaries of this committee: TH. DE DONDER (Brussels); G. KOENIGS (Paris); M. PETROWITCH (Belgrade); V. REINA (Rome).

The above information of action taken six months ago now reaches American mathematicians for the first time by indirect ways. Before the war international mathematical activities were carried on very efficiently by mathematicians and it may be hoped that they will soon resume charge of their affairs.

AT the meeting of the Edinburgh mathematical society on November 14, the following papers were read: By E. T. WHITTAKER, "A new basis for graduation"; by H. W. RICHMOND, "A geometrical proof of Morley's extension of Feuerbach's theorem."

THE two Benjamin Peirce instructorships in mathematics at Harvard University (see this BULLETIN, volume 21, page

315) are again open to general competition. Applications for the year 1920-21, accompanied by the necessary papers, should reach Professor Osgood not later than February 15, 1920, and all communications relating to these appointments should be addressed to him.

PROFESSOR E. CZUBER, of the Vienna technical school, has retired from active teaching.

At the University of Hamburg, Dr. W. BLASCHKE and Dr. E. HECKE have been appointed to professorships and Dr. J. RADON to an associate professorship of mathematics.

DR. K. KNOPP has been appointed professor of mathematics at the University of Königsberg.

At the Dresden technical school, Dr. P. E. BÖHMER has been appointed professor of the mathematics of insurance and Dr. R. VON MISES professor of the resistance of materials, hydraulics, and aerodynamics. Professor J. M. KRAUSE has retired from active teaching.

DR. R. BALDUS has been appointed professor of descriptive geometry at the Karlsruhe technical school.

DR. V. BLAESS has been appointed associate professor of mechanics at the technical school at Darmstadt.

THE following persons have recently been admitted as privat-docents in mathematics at German universities: P. BARNAYS, Miss E. NOETHER, and M. SCHMEIDLER, at the University of Göttingen; P. EPSTEIN, at the University of Frankfurt; F. LEVI, at the University of Leipzig.

THE Royal society of London has awarded a Royal medal to Mr. J. H. JEANS, for his researches in applied mathematics, and the Sylvester medal to Major P. A. MACMAHON for his researches in pure mathematics, especially in connection with the partition of numbers and analysis.

ASSISTANT professor A. F. CARPENTER, of the University of Washington, has been promoted to an associate professorship of mathematics.

PROFESSOR ARNOLD DRESDEN, secretary of the Chicago Section of the American Mathematical Society, has returned to the University of Wisconsin, after a year of service with the Red Cross in France.

At the University of California, assistant professor B. M. WOODS has been promoted to a full professorship of aerodynamics.

At the University of Oklahoma, Dr. H. C. GOSSARD, of the U. S. Naval Academy, has been appointed assistant professor of mathematics, and Mr. E. D. MEACHAM has been promoted to an assistant professorship.

ASSISTANT professor FLORENCE P. LEWIS, of Goucher College, has been promoted to an associate professorship of mathematics.

At Purdue University, assistant professor GLENN JAMES has resigned to become assistant professor of mathematics at the Carnegie Institute of Technology. Assistant professor T. E. MASON has been promoted to an associate professorship. Professor LAURENCE HADLEY, of Earlham College, has been appointed associate professor of mathematics, and Professor F. H. HODGE, of Franklin College, instructor in mathematics.

ASSOCIATE professor R. B. McCLENON, of Grinnell College, has been promoted to a full professorship of mathematics.

ASSISTANT professor C. N. MILLS, of South Dakota State College, has been appointed professor of mathematics at Heidelberg University, Tiffin, Ohio.

ADJUNCT professor J. J. LUCK has been promoted to an associate professorship of mathematics at the University of Virginia.

DR. C. H. FORSYTH, of Dartmouth College, has been promoted to an assistant professorship of mathematics.

In the mathematics department of the U. S. Naval Academy, Dr. C. C. BRAMBLE has been promoted to an assistant pro-

fessorship and Mr. PAUL HEMKE, of Northwestern University, has been appointed instructor.

DR. J. V. DEPORTE has been appointed assistant professor of mathematics at the New York State College for Teachers, Albany.

ASSISTANT professor H. H. CONWELL, of the University of Idaho, has been promoted to an associate professorship of mathematics.

DR. R. L. CHARLES has been promoted to an associate professorship of mathematics at Lehigh University.

MISS M. E. DANIELLS has been promoted to an assistant professorship of mathematics at Iowa State College.

DR. C. M. HEBBERT has been appointed instructor in mathematics at the University of Illinois.

MR. F. W. PARSONS, of Ohio Northern University, has been appointed assistant professor of mathematics and engineering in Whittier College.

DR. H. B. CURTIS and Mr. F. L. BROWN have been appointed instructors in mathematics at Northwestern University. Mr. F. L. KERR has resigned his instructorship to become registrar of the university.

MR. FREDERICK WOOD has been appointed instructor in mathematics at the University of Wisconsin.

At the College of the City of New York, Mr. HARRY LANGMAN has been appointed instructor in statistics and Dr. H. F. MACNEISH instructor in mathematics.

DR. J. W. CAMPBELL, of Wesley College, Winnipeg, has been appointed associate in mathematics and astronomy at the State University of Iowa.

DR. R. A. ARMS, of Juniata College, has been appointed instructor in mathematics at the University of Pennsylvania.

MR. J. J. TANZOLA has been appointed instructor in mathematics at Cooper Union, New York City.

THE death is reported of Professor A. GOB, of the Athénée royal at Liège, at the age of fifty-one years.

RODOLPHE GUIMARAES, author of *Les Mathématiques en Portugal*, died in 1918 at the age of fifty-two years.

THE death is reported of Professor G. DEMARTRES, of the University of Lille, at the age of seventy-one years.

DR. O. DANZER, of the Vienna technical school, died March 26, 1919.

PROFESSOR E. BÖTTCHER, of the University of Leipzig, died August 5, 1919, at the age of seventy-two years.

PROFESSOR O. DZIOBEK, of the Charlottenburg technical school, died recently at the age of sixty-three years.

PROFESSOR F. GRAEFE, of the Charlottenburg technical school, died December 2, 1918, at the age of sixty-three years.

PROFESSOR E. NETTO, of the University of Giessen, died recently at the age of seventy-two years.

DR. K. T. REYE, formerly professor at the University of Strassburg, died recently at the age of eighty-one years.

PROFESSOR R. STURM, of the University of Breslau, died recently at the age of seventy-seven years.

DR. J. WELLSTEIN, formerly professor at the University of Strassburg, died June 24, 1919, in his fiftieth year.

MR. P. E. B. JOURDAIN died October 1, 1919, at the age of thirty-eight years.

PROFESSOR JAMES MACLAY, of Columbia University, died November 28, 1919, at the age of fifty-five years.

BOOK CATALOGUES: Galloway and Porter, Cambridge, catalogue 97, list of 495 titles in mathematics and physics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- ADAMS (O. S.). General theory of polyconic projections. (U. S. Coast and Geodetic Survey, Special Publication No. 57.) Washington, Government Printing Office, 1919. 174 pp. Paper. \$0.25
- BALLANTYNE (J.). The integral calculus. On the integration of the powers of transcendental functions, new methods and theorems, calculation of the Bernoullian numbers, rectification of the logarithmic curve, integration of logarithmic binomials, etc. Boston, J. Ballantyne, 1919. 41 pp. \$1.00
- FRICKE (R.). Hauptsätze der Differential- und Integralrechnung. 6te Auflage. 1917. M. 7.60
- LAMB (H.). An elementary course of infinitesimal calculus. 3d edition revised. Cambridge, University Press, 1919. Demy 8vo. 14 + 530 pp. 20s.
- LESSER (O.). See SCHWAB (K.).
- MANGOLDT (H. VON). Einführung in die höhere Mathematik. 1ter Band. 2te Auflage. 1919. M. 24. 50
- MILNE (W. P.). Higher algebra. London, Arnold, 1919. 12 + 586 pp. 8s. 6d.
- NICOLETTI (O.). Lesioni di algebra complementare, redatte per uso degli studenti. Pisa, Bemporad, 1919. 8vo. 6 + 661 pp.
- REINHARDT (K.). Ueber die Zerlegung der Ebene in Polygone. (Diss.) Frankfurt a. M., 1918.
- SAVINO (L.). Una corrispondenza birazionale involutoria fra le rette dello spazio determinata da un fascio-schiera di quadriche. Pavia, tip. succ. Bizzoni, 1919. 8vo. 26 pp.
- SCHNEIDER (A.). See Schwab (K.).
- SCHWAB (K.) und LESSER (O.). Mathematisches Unterrichtswerk. Geometrie für Lehrer- und Lehrerinnenbildungsanstalten. 1ter Teil, von A. Schneider: Planimetrie, Grundlehren der Kegelschnitte und der analytischen Geometrie. Leipzig, Freytag, 1918. M. 3.40
- SUDRE DES CARLIÈRES (R.). L'espace inconditionné. Toulouse, Ed. Privat, 1919. 8vo. 54 pp.
- WHITEHEAD (A. N.). An enquiry concerning the principles of natural knowledge. Cambridge, University Press, 1919. 12 + 200 pp. 12s. 6d.

II. ELEMENTARY MATHEMATICS.

- AHRENS (W.). Mathematische Spiele. 4te Auflage. Leipzig, Teubner, 1919. Geb. M. 1. 90
- AMMERMAN (C.). See FORD (W. B.).
- BAFFI (C.). Elementi di algebra, ad uso della terza classe delle scuole tecniche. (Biblioteca di Scienze fisiche, matematiche e naturali.) Torino, Paravia, 1919. 16mo. 104 pp. L. 3.00

DONADT (A.). See LÜBSEN (H. B.).

FORD (W. B.) and AMMERMAN (C.). Teachers' manual, first course in algebra. New York, Macmillan, 1919. 12mo. 314 pp. \$2.00

HAWKES (H. E.), LUBY (W. A.) and TOUTON (F. C.). Complete school algebra. Revised edition. Boston, Ginn, 1919. 12mo. 9 + 507 pp. \$1.40

HJELMSLEV (J.). Stereometriske Konstruktioner. 2. Oplag. København, Gjellerup, 1919.

KILLION (R. M.). Supplemental problems in arithmetic for rural schools. Third printing, revised. (School Publication No. 4.) Los Angeles, 1919. Paper. 54 pp.

LÜBSEN (H. B.). Ausführliches Lehrbuch der Arithmetik und Algebra zum Selbstunterricht und mit Rücksicht auf die Zwecke des praktischen Lebens. Neubearbeitet von A. Donadt. 28te Auflage. 1917. M. 5.75

LUBY (W. A.). See HAWKES (H. E.).

MANNHEIMER (N.). See REINHARDT (W.).

MARTINI (Z. A.). Ripetitorio di geometria elementare, per le scuole secondarie inferiori. (Biblioteca degli Studenti.) Livorno, R. Giusti, 1919. 8 + 163 pp. L. 2.70

MORITZ (R. E.). A short course in mathematics. New York, Macmillan, 1919. 12mo. 240 pp.

PERNOT (F. E.) and WOODS (B. M.). Logarithms of hyperbolic functions to twelve significant figures. (University of California Publications in Engineering, vol. 1, No. 13.) Berkeley, University of California, 1918. Pp. 297-467. \$2.00

REINHARDT (W.), MANNHEIMER (N.) und ZEISBERG (M.). Lehr- und Übungsbuch für den mathematischen Unterricht. Ausgabe B, für Studienanstalten. 1ter, 4ter und 5ter Teil. 2te Auflage. Frankfurt a. M., Diesterweg, 1917. 145 + 202 + 219 pp. Geb.

M. 2.40 + 3.20 + 3.60

SMITH (J. H.). Algebra with answers. New York, Longmans, 1919. \$1.35

TESTI (G. M.). Primi elementi di aritmetica razionale, con numerosi esercizi, ad uso degli alunni dei ginnasi superiori. 5a edizione riveduta. Livorno, R. Giusti, 1919. 16mo. 8 + 148 pp. L. 2.00

TOUTON (F. C.). See HAWKES (H. E.).

WOODS (B. M.). See PERNOT (F. E.).

ZEISBERG (M.). See REINHARDT (W.).

III. APPLIED MATHEMATICS.

ANDRÉE (W. L.). Die Statik der Schwerlastkrane. Werft- und Schwimmkrane und Schwimmdranpontons. 1919. M. 12.00

AUERBACH (F.). Die Physik im Kriege. Eine allgemeinverständliche Darstellung der Grundlagen moderner Kriegstechnik. 4te Auflage. 1917. M. 6.25

BACH (C.). und BAUMANN (R.). Festigkeitseigenschaften und Gefügebilder der Konstruktionsmaterialien. 1915. M. 16.80

- BAUMANN (R.). See BACH (C.).
- BENISCHKE (G.). Die wissenschaftlichen Grundlagen der Elektrotechnik. 4te Auflage. 1918. M. 32.00
- BLACK (F. A.). Planetary rotation periods and group ratios: two essays on the relations between the planets in diurnal rotation and in mass. Edinburgh and London, Gall and Inglis, (1919?). 12 + 115 pp. 3s. 6d.
- CHWOLSON (O.). Lehrbuch der Physik. 1ter Band, 2te Abteilung: Die Lehre von den gasförmigen, flüssigen und festen Körpern. 2te Auflage. Braunschweig, Vieweg, 1918. Geb. M. 16.00
- DICK (C.). Leitfaden der Seemannschaft. 2te Auflage. 1916. M. 21.85
- DU PASQUIER (L. G.). Introduction à la science actuarielle. Neuchâtel, Delachaux et Niestlé, 1919. 8vo. 174 pp. Fr. 5.00
- ELGIE (J. H.). The stars by night: being the journal of a star gazer. London, Pearson, 1919. 14 + 247 pp. 1s. 6d.
- FISCHER (M.). Statik und Festigkeitslehre. Vollständiger Lehrgang zum Selbststudium für Ingenieure, Techniker und Studierende. Mit zahlreichen Beispielen und Zeichnungen. 1ter Band: Grundlagen der Statik und Berechnung vollwandiger Systeme, einschliesslich Eisenbeton. 4te Auflage. 1919. M. 26.00
- FONTVIOLENT (B. DE). Les méthodes modernes de la résistance des matériaux. Paris, Gauthier-Villars, 1919. 8vo. 82 pp.
- FÖPPL (A.). Vorlesungen über technische Mechanik. 4ter Band: Dynamik. 4te Auflage. 5ter Band: Die wichtigsten Lehren der höheren Elastizitätstheorie. 2ter Abdruck. 6ter Band: Die wichtigsten Lehren der höheren Dynamik. 2ter Abdruck. Leipzig, Teubner, 1918. M. 16.80 + 19.60 + 23.55
- G. L. See REPERTORIO.
- HAAS (A.). Einführung in die theoretische Physik mit besonderer Berücksichtigung ihrer modernen Probleme. 1ter Band. Leipzig, Veit, 1919. Geb. M. 17.50
- HARMSSEN (K.). Der Kompassflieger. 1918. M. 2.40
- IMELMAN (N. A.). Praktische Anleitungen zum Maschinenzeichnen als Grundlage zum technischen Studium. 1918. M. 5.00
- JEANS (J. H.). Problems of cosmogony and stellar dynamics. Cambridge, University Press, 1919. 8 + 293 pp. + 5 plates. 21s.
- LAMB (H.). The dynamical theory of sound. London, Arnold, 1919. 8 + 304 pp. 15s.
- LAUE (M. VON). Die Relativitätstheorie. 1ter Band: Das Relativitätsprinzip der Lorentztransformation. 3te Auflage. Braunschweig, Vieweg, 1919. Geb. M. 11.40
- MASSENZ (A.). Guida pratica del meccanico moderno: manuale teorico-pratico ad uso dei capi officina ed alunni delle scuole industriali . . . 2a edizione, rifatta ed aumentata. (Manuali Hoepli.) Milano, Hoepli, 1919. 24mo. 24 + 432 pp. L. 9.50
- MELE (D. M.). Dizionario internazionale di aeronavigazione e costruzioni aeronautiche, italiano, francese, inglese, tedesco, con indice delle quattro lingue in alfabeto unico. (Manuali Hoepli.) Milano, Hoepli, 1919. 24 mo. 7 + 227 pp. L. 6.50

- MONTANARI (C.). Elementi di geometria descrittiva. 5a edizione. (Biblioteca degli Studenti.) Livorno, R. Giusti, 1919. 16mo. 40 pp. L. 0.90
- REFEREORIO di matematiche e fisica elementari, per G. L. 12a edizione. Livorno, Giusti, 1919. 24mo. 8 + 160 pp. L. 2.50
- RÖMMLER (—). See THEBIS (R.).
- SELLENTHIN (B.). Mathematischer Leitfaden mit besonderer Berücksichtigung der Navigation. 3te Auflage. 1917. M. 11.80
- SÖNNISCHEN (T. E.). Navigation und Seemannschaft im Seeflugzeug. Ein Handbuch für Marineflieger. (Bibliothek für Luftschiffahrt und Flugtechnik, Nr. 21.) 1918. M. 7.20
- THEBIS (R.). und RÖMMLER (—). Instrumentenkunde des Fliegers. 1919. M. 4.50
- VATER (R.). Die Maschinenelemente. 3te Auflage. Leipzig, Teubner, 1919. Geb. M. 1.90
- VOGELSANG (C. W.). Die Stabilisierung der Flugzeuge. (Volckmanns Bibliothek für Flugzeuge, Nr. 9.) 1917. M. 2.45
- WEYRAUCH (R.). Hydraulisches Rechnen. Rechnungsverfahren und Zahlenwerte aus den Gebieten des Wasserbaus. 3te Auflage. 1915. M. 10.20
- WOOD, (R. W.). Researches in physical optics. Part II: Resonance radiation and resonance spectra. (Ernest Kempton Adams Fund, Publication No. 8.) New York, Columbia University Press, 1919. 4to. 8 + 184 pp. + 10 plates. \$1.50
- ZILICH (K.). Statik für Baugewerkschulen. 7te Auflage. 1ter Teil: Graphische Statik. 1919. 2ter Teil: Festigkeitslehre. 1918. 3ter Teil: Grössere Konstruktionen. 1919. M. 1.70 + 4.50 + 4.50

Publications of the American Mathematical Society

Transactions of the American Mathematical Society.

Published quarterly. Subscription price for annual volume, \$5.00; to members of the Society, \$3.75.

Mathematical Papers of the Chicago Congress, 1893.

Price, \$3.00; to members, \$1.50.

Evanston Colloquium Lectures, 1893. By FELIX KLEIN.

Price, 75 cents.

Boston Colloquium Lectures, 1903. By H. S. WHITE, F.

S. WOODS and E. B. VAN VLECK. Price, \$2.00; to members, \$1.50.

Princeton Colloquium Lectures, 1909. By G. A. BLISS

and EDWARD KASNER. Price, \$1.50; to members, \$1.00.

Madison Colloquium Lectures, 1913. By L. E. DICKSON

and W. F. OSGOOD. Price, \$2.00; to members, \$1.50.

Cambridge Colloquium Lectures, 1916. Part I. By G.

C. EVANS. Price, \$1.50; to members, \$1.00.

**Address all orders to AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.**

Whole No. 284

CONTENTS

	PAGE
The October Meeting of the American Mathematical Society. By Professor F. N. COLE - - - - -	145
The October Meeting of the San Francisco Section. By Professor B. A. BERNSTEIN - - - - -	152
On the Proof of Cauchy's Integral Formula by Means of Green's Formula. By Mr. J. L. WALSH - - - - -	155
A Set of Completely Independent Postulates for the Linear Order η . By Professor M. G. GABA - - - - -	158
Certain Properties of Binomial Coefficients. By Professor W. D. CAIRNS - - - - -	160
The Work of Poincaré on Automorphic Functions. By Pro- fessor G. D. BIRKHOFF - - - - -	164
A Brief Account of the Life and Work of the Late Professor Ulisse Dini. By Professor W. B. FORD - - - - -	173
Shorter Notices - - - - -	177
Notes - - - - -	184
New Publications - - - - -	189

Changes of address of members, exchanges, and subscribers should be communicated at once to the Secretary of the American Mathematical Society, 501 West 116th Street, New York.

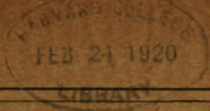
Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to The American Mathematical Society, 41 North Queen St., Lancaster, Pa., or 501 West 116th Street, New York.

The initiation fees and annual dues of members of the American Mathematical Society are payable to the Treasurer of the American Mathematical Society, Professor J. H. TANNER, Cornell University, Ithaca, N. Y.

Articles for insertion in the BULLETIN should be addressed to the
BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.

The following dates have been fixed for the meetings of the Society :

New York: February 28, 1920. April 24, 1920



BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

EDITED BY

F. N. COLE

VIRGIL SNYDER

J. W. YOUNG

ALEXANDER ZIWET

D. E. SMITH

T. LEVI-CIVITA

R. C. ARCHIBALD

VOLUME XXVI., NUMBER 5

FEBRUARY, 1920

PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK

1920

Entered at the Post Office at Lancaster, Pa., as second class matter
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER

Whole No. 285

Digitized by Google \$5.00 a year

Publications of the American Mathematical Society

Transactions of the American Mathematical Society.

Published quarterly. Subscription price for annual volume, \$5.00; to members of the Society, \$3.75.

Mathematical Papers of the Chicago Congress, 1893.

Price, \$3.00; to members, \$1.50.

Evanston Colloquium Lectures, 1893. By FELIX KLEIN.

Price, 75 cents.

Boston Colloquium Lectures, 1903. By H. S. WHITE, F.

S. WOODS and E. B. VAN VLECK. Price, \$2.00; to members, \$1.50.

Princeton Colloquium Lectures, 1909. By G. A. BLISS

and EDWARD KASNER. Price, \$1.50; to members, \$1.00.

Madison Colloquium Lectures, 1913. By L. E. DICKSON

and W. F. OSGOOD. Price, \$2.00; to members, \$1.50.

Cambridge Colloquium Lectures, 1916. Part I. By G.

C. EVANS. Price, \$1.50; to members, \$1.00.

**Address all orders to AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.**

INTEGRO-DIFFERENTIAL EQUATIONS WITH CONSTANT LIMITS OF INTEGRATION.

BY DR. I. A. BARNETT.

(Read before the American Mathematical Society December 28, 1918.)

CONSIDER the linear integro-differential equation

$$(1) \quad \frac{\partial}{\partial \tau} u(\xi, \tau) = \int_a^b K(\xi, \eta) u(\eta, \tau) d\eta,$$

where τ is a real variable ranging over $|\tau - \tau_0| \leq c$, and $K(\xi, \eta)$ is a continuous function defined in the square $a \leq \xi \leq b$, $a \leq \eta \leq b$. Volterra has shown* that the most general solution of this equation, reducing for $\tau = \tau_0$ to the arbitrary continuous function $u_0(\xi)$, is given by

$$(2) \quad u(\xi, \tau) = u_0(\xi) + \int_a^b L(\xi, \eta, \tau) u_0(\eta) d\eta,$$

where

$$L(\xi, \eta, \tau) = \sum_{n=1}^{\infty} \frac{(\tau - \tau_0)^n}{n!} K_n(\xi, \eta),$$

the functions K_n denoting, as usually, the iterated kernels of K with the understanding that $K_1 = K$.

It is proposed in this paper to give a solution of equation (1) in terms of the characteristic numbers and functions of the kernel K for the cases in which K is symmetric and skew-symmetric. Furthermore, some extensions of the theory to more general kernels will be pointed out and a formal analogy between integro-differential equations and partial differential equations will be shown. It is of interest to note that equation (1) may be considered as the limiting case of the finite system of differential equations with constant coefficients

$$\frac{du_i}{d\tau} = \sum_{j=1}^n K_{ij} u_j \quad (i = 1, \dots, n),$$

and the results for the integro-differential equations could be predicted from the well-known theory of this last system.†

* Volterra, *Lincei Rendiconti*, serie 5 (1914), 2d semester, p. 394.

† Schlesinger, "Zur Theorie der linearen Integrodifferentialgleichungen," *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 24 (1915-16), p. 84.

§ 1. *The Symmetric Kernel.*

THEOREM. *Through the element (τ_0, u_0) , where u_0 is a real continuous function of ξ , there exists one and but one solution of equation (1), viz.,*

$$(3) \quad v(\xi, \tau) = u_0(\xi) + \sum_{n=1}^{\infty} \varphi_n(\xi) (e^{(\tau-\tau_0)/\lambda_n} - 1) \int_a^b \varphi_n(\eta) u_0(\eta) d\eta,$$

where the λ_n are the characteristic numbers of the kernel K (necessarily real) and the φ_n are the characteristic functions.

The λ_n are arranged, as usually, in the order of their increasing absolute values, and the φ_n are supposed normalized and orthogonal.

Two methods of proof suggest themselves. One could start with Volterra's solution (2), and obtain the form (3), by making use of the well known properties concerning iterated kernels of a symmetric kernel.* A better method, however, is to proceed directly.

Consider the infinite sum

$$S(\xi, \tau) = \sum_{n=1}^{\infty} \varphi_n(\xi) (e^{(\tau-\tau_0)/\lambda_n} - 1) \int_a^b \varphi_n(\eta) u_0(\eta) d\eta.$$

One may write

$$|S(\xi, \tau)| \leq \sum_{n=1}^{\infty} \left| \frac{\varphi_n(\xi)}{\lambda_n} \int_a^b \varphi_n(\eta) u_0(\eta) d\eta \right| |\tau - \tau_0| \\ \times \left| 1 + \frac{1}{2!} \left(\frac{\tau - \tau_0}{\lambda_n} \right) + \frac{1}{3!} \left(\frac{\tau - \tau_0}{\lambda_n} \right)^2 + \dots \right|.$$

Hence, since the λ_n are bounded from zero, i.e., $|\lambda_n| > \alpha > 0$ for all n , it follows that

$$|\tau - \tau_0| \left| 1 + \frac{1}{2!} \left(\frac{\tau - \tau_0}{\lambda_n} \right) + \frac{1}{3!} \left(\frac{\tau - \tau_0}{\lambda_n} \right)^2 + \dots \right|$$

has an upper bound M for all τ 's for which $|\tau - \tau_0| \leq c$. Furthermore, the infinite series

$$\sum_{n=1}^{\infty} \frac{\varphi_n(\xi)}{\lambda_n} \int_a^b \varphi_n(\eta) u_0(\eta) d\eta,$$

converges absolutely uniformly in the interval $a \leq \xi \leq b$, by

* G. Vivanti, *Elementi della Teoria delle Equazioni integrali*, § 61, § 71, § 76.

the Hilbert-Schmidt expansion theorem.* Thus, the series representing $S(\xi, \tau)$ converges uniformly for $a \leq \xi \leq b$, $|\tau - \tau_0| \leq c$, so that $S(\xi, \tau)$ is a continuous function for all (ξ, τ) lying in this same region.

Differentiating $S(\xi, \tau)$ with respect to τ , one obtains

$$\sum_{n=1}^{\infty} \frac{\varphi_n(\xi)}{\lambda_n} e^{(\tau-\tau_0)/\lambda_n} \int_a^b \varphi_n(\eta) u_0(\eta) d\eta,$$

which is dominated by*

$$e^{|\tau-\tau_0|/a} \int_a^b K(\xi, \eta) u_0(\eta) d\eta,$$

from which it follows that $v(\xi, \tau)$ has a derivative which converges uniformly for all τ 's of $|\tau - \tau_0| \leq c$.

Substituting (3) in (1), one finds

$$(4) \quad \sum_{n=1}^{\infty} \frac{\varphi_n(\xi)}{\lambda_n} e^{(\tau-\tau_0)/\lambda_n} \int_a^b \varphi_n(\eta) u_0(\eta) d\eta = \int_a^b K(\xi, \eta) u_0(\eta) d\eta \\ + \int_a^b K(\xi, \eta) \sum_{n=1}^{\infty} \varphi_n(\eta) d\eta \int_a^b \varphi_n(\zeta) u_0(\zeta) d\zeta (e^{(\tau-\tau_0)/\lambda_n} - 1).$$

Since the second term on the right-hand side converges uniformly, one may integrate term by term, after which the right hand side reduces to

$$\int_a^b K(\xi, \eta) u_0(\eta) d\eta + \sum_{n=1}^{\infty} \frac{\varphi_n(\xi)}{\lambda_n} e^{(\tau-\tau_0)/\lambda_n} \int_a^b \varphi_n(\eta) u_0(\eta) d\eta \\ - \sum_{n=1}^{\infty} \frac{\varphi_n(\xi)}{\lambda_n} \int_a^b \varphi_n(\eta) u_0(\eta) d\eta,$$

which is simply the left-hand side of (4), by the Hilbert-Schmidt theorem. Hence, (3) is a solution of (1).

The uniqueness of the solution may be shown as follows: Equation (1) with the initial condition is evidently equivalent to the integral equation

$$u(\xi, \tau) = \int_{\tau_0}^{\tau} \int_a^b K(\xi, \eta) u(\eta, \tau) d\eta d\tau + u_0(\xi).$$

Suppose there could be another solution $\bar{u}(\xi, \tau)$ with the same

* Vivanti, § 89.

properties as $u(\xi, \tau)$. Then

$$\bar{u}(\xi, \tau) = \int_{\tau_0}^{\tau} \int_a^b K(\xi, \eta) \bar{u}(\eta, \tau) d\eta d\tau + u_0(\xi).$$

Hence,

$$\bar{u}(\xi, \tau) - u(\xi, \tau) = \int_{\tau_0}^{\tau} \int_a^b K(\xi, \eta) [\bar{u}(\eta, \tau) - u(\eta, \tau)] d\eta d\tau,$$

and calling U the maximum of $|\bar{u}(\eta, \tau) - u(\eta, \tau)|$ and K the maximum of $K(\xi, \eta)$, in their respective regions of definitions, one may write

$$|\bar{u}(\xi, \tau) - u(\xi, \tau)| \leq KU |\tau - \tau_0| |b - a|,$$

and by continued application of this inequality, one finally arrives at the relation

$$|\bar{u}(\xi, \tau) - u(\xi, \tau)| \leq [KU(b-a)]^n \frac{|\tau - \tau_0|^n}{n!},$$

from which, by a passage to the limit as $n \rightarrow \infty$, one obtains the desired result.

COROLLARY 1. *If the initial function $u_0(\xi)$ has the form $\int_a^b K(\xi, \eta) \rho(\eta) d\eta$, where $\rho(\eta)$ is real and continuous in $a \leq \eta \leq b$, then the solution (3) may be put in the form*

$$\begin{aligned} (5) \quad v(\xi, \tau) &= \sum_{n=1}^{\infty} \frac{\varphi_n(\xi)}{\lambda_n} e^{(\tau-\tau_0)/\lambda_n} \int_a^b \varphi_n(\eta) \rho(\eta) d\eta \\ &= \sum_{n=1}^{\infty} \varphi_n(\xi) e^{(\tau-\tau_0)/\lambda_n} \int_a^b \varphi_n(\eta) u_0(\eta) d\eta. \end{aligned}$$

This follows immediately upon substitution of $\int_a^b K(\xi, \eta) \rho(\eta) d\eta$ for $u_0(\xi)$ in (3) and an application of the Hilbert-Schmidt expansion.

COROLLARY 2. *The series in the solution (3) will terminate, i.e., will have the form*

$$(6) \quad u_0(\xi) + \sum_{n=1}^m \varphi_n(\xi) (e^{(\tau-\tau_0)/\lambda_n} - 1) \int_a^b \varphi_n(\eta) u_0(\eta) d\eta,$$

when, and only when, the kernel K can be written in the form

$$K(\xi, \eta) = \sum_{n=1}^m \frac{\varphi_n(\xi) \varphi_n(\eta)}{\lambda_n}.$$

It is understood, of course, that corresponding to each root of multiplicity p there will be p terms in (6).

This is an immediate consequence of a well-known result in the theory of integral equations.* For example, the solution of the integro-differential equation

$$\frac{\partial}{\partial \tau} u(\xi, \tau) = \int_0^{\pi/2} \xi \eta u(\eta, \tau) d\eta$$

passing through τ_0 and $u_0(\xi) = \int_0^{\pi/2} \xi \eta \sin \eta d\eta = \xi$ is given by $\xi e^{\pi^2(\tau-\tau_0)/24}$, since the only normed characteristic function is $\frac{2}{\pi} \sqrt{\frac{6}{\pi}} \xi$ with the characteristic number $24/\pi^3$.

§ 2. The Skew-Symmetric Kernel.

If it is now supposed that $K(\xi, \eta) = -K(\eta, \xi)$, the corresponding theorem is as follows:

THEOREM. *Through the element (τ_0, u_0) there exists one and but one solution*

$$(7) \quad v(\xi, \tau) = u_0(\xi) + \frac{1}{2} \sum_{n=1}^{\infty} [\bar{d}_n \varphi_n(\xi) (e^{(\tau-\tau_0)/\lambda_n} - 1) + d_n \bar{\varphi}_n(\xi) (e^{(\tau-\tau_0)/\bar{\lambda}_n} - 1)],$$

reducing for $\tau = \tau_0$ to $u_0(\xi)$, where

$$d_n = \int_a^b \varphi_n(\eta) u_0(\eta) d\eta,$$

and where $\bar{d}_n$, $\bar{\lambda}_n$, and $\bar{\varphi}_n(\xi)$ are the conjugates of d_n , λ_n , and $\varphi_n(\xi)$ respectively.

Here the λ_n , $\bar{\lambda}_n$ are the characteristic numbers and the $\varphi_n(\xi)$, $\bar{\varphi}_n(\xi)$ are the characteristic functions which are supposed normed and biorthogonalized. The proof of this theorem is like that of the preceding one and will not be given. It suffices to remark that in this case use is made of a development analogous to that of Hilbert-Schmidt for the symmetric case.†

* Lalesco, Introduction à la Théorie des Equations intégrales (1912), p. 66.

† Lalesco, p. 77.

By making use of the particular form of the characteristic numbers, one may put (7) into a somewhat different form. It is well-known that the characteristic numbers of a skew-symmetric kernel are of the form $\pm \nu i$ and that if νi is a characteristic number with $\varphi(\xi)$ as characteristic function, then $-\nu i$ is also a characteristic number with $\bar{\varphi}(\xi)$ as corresponding characteristic function. If in (7) one introduces the notations

$$\begin{aligned}\lambda_n &= \nu_n i, & \varphi_n(\xi) &= \psi_n(\xi) + i\chi_n(\xi), \\ e_n &= \int_a^b \psi_n(\eta) u_0(\eta) d\eta, & f_n &= \int_a^b \chi_n(\eta) u_0(\eta) d\eta,\end{aligned}$$

one finds, after some reductions, that it may be written

$$(8) \quad \begin{aligned}v(\xi, \tau) = u_0(\xi) + \sum_{n=1}^{\infty} \left[\{e_n \psi_n(\xi) + f_n \chi_n(\xi)\} \left(\cos \frac{\tau - \tau_0}{\nu_n} - 1 \right) \right. \\ \left. + \{e_n \chi_n(\xi) - f_n \psi_n(\xi)\} \sin \frac{\tau - \tau_0}{\nu_n} \right].\end{aligned}$$

Just as in the symmetric case, it is to be remarked here that when $u_0(\xi)$ has the special form $\int_a^b K(\xi, \eta) \rho(\eta) d\eta$, where $\rho(\eta)$ is a real continuous function, then (7) reduces to

$$(9) \quad v(\xi, \tau) = \sum_{n=1}^{\infty} \left[\bar{g}_n \frac{\varphi_n(\xi)}{\lambda_n} e^{(\tau - \tau_0)/\lambda_n} + g_n \frac{\bar{\varphi}_n(\xi)}{\bar{\lambda}_n} e^{(\tau - \tau_0)/\bar{\lambda}_n} \right],$$

where

$$g_n = \int_a^b \varphi_n(\xi) \rho(\xi) d\xi$$

and $\bar{g}_n$ is the conjugate of g . Moreover, (8) becomes

$$(10) \quad v(\xi, \tau) = \sum_{n=1}^{\infty} \frac{2}{\nu_n} \left[\beta_n(\xi) \cos \frac{\tau - \tau_0}{\nu_n} + \gamma_n(\xi) \sin \frac{\tau - \tau_0}{\nu_n} \right],$$

where

$$\begin{aligned}\beta_n(\xi) &= h_n \chi_n(\xi) - k_n \psi_n(\xi), & \gamma_n(\xi) &= h_n \psi_n(\xi) + k_n \chi_n(\xi), \\ h_n &= \int_a^b \psi_n(\xi) \rho(\xi) d\xi, & k_n &= \int_a^b \chi_n(\xi) \rho(\xi) d\xi.\end{aligned}$$

Finally, in view of the fact* that every skew-symmetric kernel

* Lalesco, p. 77.

having a finite number of characteristic numbers is necessarily of the form

$$\sum_{n=1}^m \left(\frac{\varphi_n(\xi) \bar{\varphi}_n(\eta)}{\nu_n i} + \frac{\bar{\varphi}_n(\xi) \varphi_n(\eta)}{-\nu_n i} \right),$$

it follows that for such kernels and only for such, the solution (7) has a finite number of terms.

As an example of an integro-differential equation of this last type, consider the equation

$$(11) \quad \frac{\partial}{\partial \tau} u(\xi, \tau) = \int_0^{2\pi} \sin \sigma(\xi - \eta) u(\eta, \tau) d\eta,$$

where σ is an odd integer. Now, since

$$\sin \sigma(\xi - \eta) = \frac{e^{\sigma i \xi} e^{-\sigma i \eta}}{\sqrt{2\pi} \sqrt{2\pi}} + \frac{e^{-\sigma i \xi} e^{\sigma i \eta}}{\sqrt{2\pi} \sqrt{2\pi}},$$

it is clear that the only characteristic numbers of the kernel $\sin \sigma(\xi - \eta)$ are i/π and $-i/\pi$ with the corresponding normed biorthogonal characteristic functions $e^{\sigma i \xi / \sqrt{2\pi}}$, $e^{-\sigma i \xi / \sqrt{2\pi}}$. Hence, the solution of (11), which reduces for $\tau = \tau_0$ to an arbitrary continuous function $u_0(\xi)$, is, according to (7)

$$v(\xi, \tau) = u_0(\xi) + \frac{1}{2\pi} [e^{-\sigma i \xi} (e^{-\pi i (\tau - \tau_0)} - 1)] \int_0^{2\pi} e^{\sigma i \eta} u_0(\eta) d\eta \\ + e^{\sigma i \xi} (e^{\pi i (\tau - \tau_0)} - 1) \int_0^{2\pi} e^{-\sigma i \eta} u_0(\eta) d\eta,$$

or, by formula (8)

$$v(\xi, \tau) = u_0(\xi) - \frac{2}{\pi} \int_0^{2\pi} \sin \frac{\pi(\tau - \tau_0)}{2} \sin \left[\frac{\pi(\tau - \tau_0)}{2} - \sigma(\xi - \eta) \right] u_0(\eta) d\eta.$$

§ 3. Some Extensions and Applications of the Preceding Results.

In recalling the proofs of formulas (3) and (7), one sees that the essential feature was the fact that the symmetric and skew-symmetric kernels had an expansion (Hilbert-Schmidt) associated with them. This is also true for the symmetrizable kernels (Marty). A kernel $K(\xi, \eta)$ is said

to be symmetrizable if there exists a definite symmetric kernel $G(\xi, \eta)$ such that either

$$P(\xi, \eta) = \int_a^b G(\xi, \zeta) K(\zeta, \eta) d\zeta$$

or

$$Q(\xi, \eta) = \int_a^b K(\xi, \zeta) G(\zeta, \eta) d\zeta$$

is symmetric.* As in the symmetric case, the characteristic numbers λ_n are real and the theorem on the development says that, if $f(\xi)$ is of the form $\int_a^b K(\xi, \eta) h(\eta) d\eta$, then†

$$f(\xi) = \sum_{n=1}^{\infty} \frac{\int \int G(\eta, \zeta) \varphi_n(\zeta) f(\eta) d\eta d\zeta}{\lambda_n} \varphi_n(\xi),$$

where the $\varphi_n(\xi)$ are the characteristic functions of $K(\xi, \eta)$.‡ One could then easily write down the form of the solution corresponding to (3). It is to be remarked that both the symmetric and skew-symmetric kernels are special instances of symmetrizable kernels, as well as the kernels $A(\xi)G(\xi, \eta)$, $A(\xi)G(\xi, \eta)B(\eta)$ and $\int_a^b H(\xi, \eta)G(\zeta, \eta)d\zeta$, where G is a definite symmetric kernel and H is symmetric.

One could also generalize farther by considering equations of the type

$$\frac{\partial}{\partial \tau} u(p, \tau) = J_{qr} K(p, q) u(r, \tau),$$

where p, q , and r have arbitrary ranges§; K is hermitian, i.e., $\bar{K}(p, q) = K(q, p)$; and J is a linear functional operation. In view of Moore's generalization of the Hilbert-Schmidt theory, one could easily generalize the results of Section 1.

It would be interesting to see how to express the solution of the integro-differential equation (1) in terms of the characteristic functions and numbers when the kernel is perfectly general, as in Volterra's case. For this purpose it would

* Lalesco, p. 78.

† Lalesco, p. 84.

‡ The functions $\varphi_n(\xi)$ and $\psi_n(\xi) = \int G(\xi, \eta) \varphi_n(\eta) d\eta$ are supposed normed and biorthogonalized.

§ E. H. Moore, "On the foundations of the theory of linear integral equations," *BULLETIN*, vol. 18 (1912), pp. 334-362. Hildebrandt, "On a theory of linear differential equations in general analysis," *Transactions Amer. Math. Society*, vol. 18 (1917), pp. 73-96.

seem that one would have to consider the elementary divisors of the Fredholm determinant, as in the theory of the finite system of differential equations with constant coefficients.

Before leaving this subject, one should notice that the integro-differential equation

$$(12) \quad \frac{\partial}{\partial \tau} u(\xi, \tau) = y(\tau)u(\xi, \tau) + \int_a^b K(\xi, \eta)u(\eta, \tau)d\eta,$$

where $y(\tau)$ is a continuous function of τ in the interval $|\tau - \tau_0| \leq c$ and $K(\xi, \eta)$ is, for example, symmetric, is readily reducible to an equation of type (1). For, multiplying both sides of (12) by $e^{\int_{\tau_0}^{\tau} y(\tau)d\tau}$ and replacing $e^{\int_{\tau_0}^{\tau} y(\tau)d\tau} u(\xi, \tau)$ by $w(\xi, \tau)$, one finds the equation

$$\frac{\partial}{\partial \tau} w(\xi, \tau) = \int_a^b K(\xi, \eta)w(\eta, \tau)d\eta,$$

so that the solution of (12), having an initial function $u_0(\xi)$, is given by

$$(13) \quad \begin{aligned} v(\xi, \tau) = u_0(\xi)e^{-\int_{\tau_0}^{\tau} y(\tau)d\tau} + \sum_{n=1}^{\infty} \varphi_n(\xi)(e^{(\tau-\tau_0)/\lambda_n} - 1) \\ \times e^{-\int_{\tau_0}^{\tau} y(\tau)d\tau} \int_a^b \varphi_n(\eta)u_0(\eta)d\eta. \end{aligned}$$

In particular, if $y(\tau) \equiv 1$, this becomes

$$(14) \quad \begin{aligned} v(\xi, \tau) = e^{\tau_0-\tau}u_0(\xi) \\ + \sum_{n=1}^{\infty} \varphi_n(\xi)(e^{(\tau-\tau_0)(1-\lambda_n)/\lambda_n} - e^{\tau_0-\tau}) \int_a^b \varphi_n(\eta)u_0(\eta)d\eta. \end{aligned}$$

Consider also the non-homogeneous equation

$$(15) \quad \frac{\partial}{\partial \tau} u(\xi, \tau) = \int_a^b K(\xi, \eta)u(\eta, \tau)d\eta + z(\xi),$$

where $z(\xi)$ is continuous and has the form $\int_a^b K(\xi, \eta)x(\eta)d\eta$.

Then it can be readily verified that the unique solution passing through $u_0(\xi)$ is given by

$$(16) \quad \begin{aligned} v(\xi, \tau) = u_0(\xi) + \sum_{n=1}^{\infty} \varphi_n(\xi)(e^{(\tau-\tau_0)/\lambda_n} - 1) \\ \int_a^b \varphi_n(\eta)[u_0(\eta) + z(\eta)]d\eta, \end{aligned}$$

which obviously reduces to (3) when $z(\eta) = 0$.

It is now desired to point out an analogy between integro-differential equations and partial differential equations. Consider the problem of the free vibrations of an elastic string. Let the string when stretched be of length unity and of homogeneous density. Moreover, let the initial position of the string be $u_0(\xi)$ and let each particle start its motion with the initial velocity zero. Then, it can be shown that, if the units are chosen properly, the differential equation of the problem will be given by*

$$(17) \quad \frac{\partial^2}{\partial \tau^2} u(\xi, \tau) = c^2 \frac{\partial^2}{\partial \xi^2} u(\xi, \tau), \quad (c = \text{constant}),$$

$$u(\xi, \tau_0) = u_0(\xi), \quad u(0, \tau) = u(1, \tau) = 0.$$

By following the usual methods for solving such systems, one obtains that a solution of (17) is given by

$$(18) \quad u(x, \tau) = 2 \sum_{n=1}^{\infty} \sin n\pi x e^{n^2\pi^2(\tau-\tau_0)} \int_0^1 u_0(\xi) \sin n\pi \xi d\xi,$$

if it is assumed that $u_0(\xi)$ is of such a character that it is expandible in a Fourier series.

Now consider the integro-differential equation

$$(19) \quad \frac{\partial}{\partial \tau} u(\xi, \tau) = \int_0^1 K(\xi, \eta) u(\eta) d\eta, \quad u(\xi, \tau_0) = u_0(\xi),$$

where

$$K(\xi, \eta) = \begin{cases} (1-\eta)\xi, & 0 \leq \xi \leq \eta \\ \eta(1-\xi), & \eta \leq \xi \leq 1 \end{cases}$$

is the Green's function relative to the system (17).

It is well known that the characteristic numbers of this kernel have the form $n^2\pi^2$ with the normed orthogonal characteristic functions $\sqrt{2} \sin n\pi\xi$. So that, if one supposes that $u_0(\xi)$ has the form $\int_0^1 K(\xi, \eta) \rho(\eta) d\eta$ and hence is expandible in a Fourier series, one obtains immediately from (5) that the general solution of (19) is given by

$$(20) \quad u(x, \tau) = 2 \sum_{n=1}^{\infty} \sin n\pi x e^{(\tau-\tau_0)/n^2\pi^2} \int_0^1 u_0(\xi) \sin n\pi \xi d\xi,$$

* Weber-Riemann, Lehrbuch der partiellen Differential-Gleichungen, § 83.

which is seen to agree with (18), except for the fact that the coefficient of $\tau - \tau_0$ in the exponent of e in (20) is the reciprocal of that in (19). This brings out the relation between systems (19) and (17). This is readily generalized to all systems like (17) which lead to self-adjoint differential equations of the second order with boundary conditions.

HARVARD UNIVERSITY,
October 21, 1919.

ON A PENCIL OF NODAL CUBICS.

BY PROFESSOR NATHAN ALTSHILLER-COURT.

(Read before the American Mathematical Society December 31, 1919.)

CONSIDER a pencil Γ of nodal cubics having in common three collinear points, the double point, and the two tangents at this point.

1. Let Γ_n be one of the cubics of the pencil Γ . A variable secant passing through one of the three basic collinear points A, B, C , say A , cuts Γ_n in pairs of points which are projected from the double point O by an involution of rays, the two tangents OT_1, OT_2 (T_1, T_2 are points of the basic line ABC) to Γ_n at O being a pair of conjugate elements in this involution.¹ The lines OB, OC , are obviously another pair of conjugate elements in this involution. The double lines OM_n, OM_n' , of this involution project from O the two points of contact M_n, M_n' of the two tangents from A to Γ_n .

When Γ_n describes the pencil Γ the two pairs of lines OT_1, OT_2 and OB, OC , remain fixed, by hypothesis, hence the involution

$$(I) \quad O(T_1T_2, BC, M_nM_n, M_n'M_n')$$

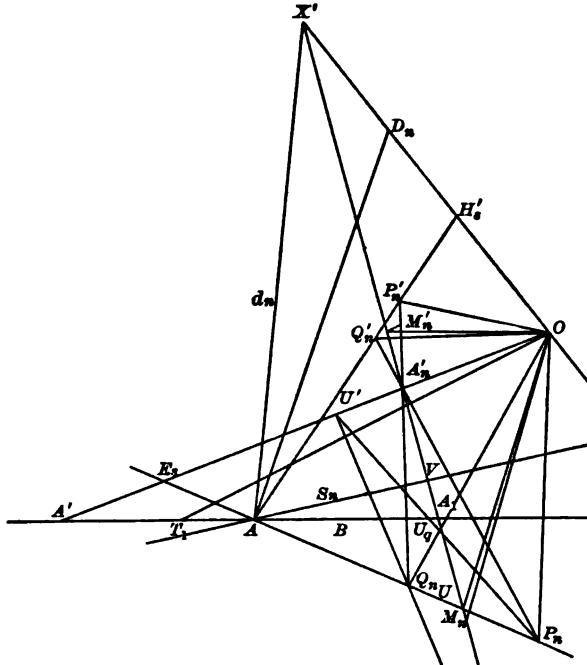
is fixed, and therefore also its doubles lines OM_n, OM_n' . Consequently: *The points of contact of the pairs of tangents from one of the basic points to curves of the pencil Γ lie on two fixed lines passing through the double point.*

2. The corresponding point A_n' of A on Γ_n is projected

¹ M. Chasles, "Mémoire sur la construction des racines des équations du troisième et du quatrième degré," *Comptes Rendus de l'Académie des Sciences*, tome 41, p. 677. E. de Jonquières, *Mélanges de Géométrie pure*, 1856, p. 180.

from O by a line OA_n' which is the harmonic conjugate of OA with respect to OT_1, OT_2 .²

When Γ_n varies the three lines OA, OT_1, OT_2 remain fixed, by hypothesis, therefore: *The corresponding points on the*



cubics of the pencil Γ of any one of the basic points lie on a straight line passing through the double point.

3. The chord of contact M_nM_n' of the two tangents AM_n, AM_n' from A to Γ_n (1) passes through the corresponding point A_n' (2) of A on Γ_n .³ The two lines $OA_n', A_n'M_nM_n'$ meet the line ABC in two points A', A_1 harmonically separated by B, C , according to a theorem of McLaurin.⁴

When Γ_n varies the point A' remains fixed (2), and since B, C are also fixed, hence A_1 is a fixed point. Thus: *The chords of contact of the pairs of tangents from one of the basic*

² See, for instance, W. Binder, *Theorie der unicursalen Plancurven*, p. 293.

³ de Jonquières, loc. cit., p. 226, prop. VII.

⁴ de Jonquières, loc. cit., p. 237, prop. XIV.

points to the cubics of the pencil Γ pass through a fixed point of the basic line.

4. Let D_n be the tangential of A on Γ_n , i.e., the point where Γ_n is met again by the tangent to this cubic at the point A . In the involution (I) (1) to the line OD_n corresponds the line OA . Since neither this involution, nor the line OA vary with Γ_n , the line OD_n is therefore fixed. Consequently: *The tangentials of any of the basic points on the cubics of the pencil Γ lie on a straight line passing through the double point.*

5. Let P_n be any point on Γ_n . The line AP_n meets Γ_n again in Q_n , and the two lines OP_n , OQ_n are conjugate in the involution of rays (I) (1).

When P_n varies with Γ_n on the line OP_n , the line OQ_n , being the conjugate of the fixed line OP_n in (I), will remain fixed. Hence: *The lines projecting from a basic point the range of points determined by the pencil Γ on any line passing through the double point, meet the respective cubics again on a fixed straight line passing through the double point.*

5a. The corresponding points of A describe a straight line (2), hence (5): *The lines joining a basic point to its corresponding points on the cubics of the pencil Γ meet the respective curves again on a straight line passing through the double point.*

6. The pairs of points determined on Γ_n by a variable secant passing through P_n are projected from O by an involution of rays in which OT_1 , OT_2 are a couple of conjugate elements, and the lines OA , OQ_n (5) are another.¹ In this involution to the ray OP_n corresponds the line OL_n projecting from O the tangential L_n of P_n on Γ_n .⁵

If the line OP_n is maintained fixed while Γ_n describes the pencil Γ , the point Q_n will describe the range of points OQ_n (5). Thus when Γ_n varies the involution of rays just considered has besides the fixed couple OT_1 , OT_2 also the fixed couple OA , OQ_n , hence this involution is fixed, and the conjugate OL_n in this involution of the fixed ray OP_n is also fixed. Consequently: *The tangentials of the range of points determined by the curves of the pencil Γ on a line passing through the double point, lie on a straight line also passing through the double point.*

6a. Since L_n describes a line passing through O , its tangentials, which are the second tangentials of P_n , will also describe a straight line. In turn we may consider the tan-

⁵ Binder, loc. cit.

gentials of these tangentials, etc. Thus: *The tangentials of any given order of the range of points determined by the pencil Γ on a line passing through the double point, lie on a straight line also passing through the double point.*

6b. We have seen that the tangentials of A describe a straight line (4), hence (6a): *The tangentials of any given order of a basic point on the cubics of the pencil Γ lie on a straight line passing through the double point.*

7. The double elements OT_n, OT_n' of the involution of rays $O(T_1T_2, AQ_n)$ (6) project from O the points of contact T_n, T_n' of the tangents P_nT_n, P_nT_n' from P_n to Γ_n . Since this involution remains fixed when P_n varies with Γ on the line OP_n (6), we have: *The points of contact of the pairs of tangents drawn to the cubics of the pencil Γ from the points which these curves respectively determine on a line passing through the double point, lie on two straight lines passing through the double point.*

8. The lines $OA_n', A_n'A_1$ (3) meet the secant AP_nQ_n (6) in two points harmonically separated by P_n and Q_n .⁴ Let $U \equiv (AP_nQ_n, A_1A_n')$.

Let R_n be any other point of Γ_n , and S_n the third point of intersection of Γ_n with AR_n . The lines $OA_n', A_n'A_1$ (3) determine on the secant AR_nS_n two points harmonically separated by R_n, S_n .⁴ Let $V \equiv (R_nS_n, A_1A_n')$.

If the lines OP_n, OR_n are kept fixed when Γ_n describes the pencil Γ , the points P_n, R_n will describe two ranges of points. The points Q_n, S_n will describe two other ranges of points on the lines OQ_n, OS_n (5). Hence the point U will also describe a straight line, namely the harmonic conjugate of the fixed line OA_n' (2) with respect to the couple of lines OP_n, OQ_n . Similarly V will describe the harmonic conjugate of OA_n' with respect to OR_n, OS_n . Thus we have

$$(P_n \dots) \propto A(P_n \dots) \propto A_1(U \dots) \\ \propto (V \dots) \propto A(V \dots) \propto (R_n \dots).$$

It follows immediately from this construction that (i) when P_n coincides with O , the point R_n likewise coincides with O , and (ii) when P_n coincides with the point (OP_n, ABC) , the point R_n coincides with (OR_n, ABC) . Hence: *The lines joining the pairs of points determined by a variable cubic of the pencil Γ on two fixed lines passing through the double point pass through a fixed point of the basic line.*

9. The tangent $P_n L_n$ to Γ_n at P_n joins two points which Γ_n determines on the two lines OP_n (5), OL_n (6). Consequently (8): *The tangents to the curves of the pencil Γ at their points of intersection with a line passing through the double point, pass through a fixed point of the basic line.*

9a. This proposition (9) may be applied to the range of points A_n' (2) and also to the range of points D_n (4). The reader may formulate the resulting theorems.

10. The tangents $P_n T_n$, $P_n T_n'$ from P_n to Γ_n have their points of contact T_n , T_n' on two fixed straight lines (7). Each of these two tangents passes through a fixed point of the basic line, when Γ_n varies (9), hence we have, relatively to the variable point P_n of the range OP_n : *The pairs of tangents drawn to the cubics of the pencil Γ from the points which these curves respectively determine on a fixed line passing through the double point, pass through a pair of fixed points of the basic line.*

11. The corresponding point P_n' of P_n on Γ_n lies on the line OP_n' which is harmonically separated from OP_n by the tangents OT_1 , OT_2 to Γ_n at O ,² therefore when P_n varies with Γ_n on OP_n , the line OP_n' remains fixed. Thus: *A line through the double point meets the cubics of the pencil Γ in a range of points the corresponding points of which on the respective curves lie on a straight line passing through the double point; and by virtue of (8): The lines joining the pairs of corresponding points thus obtained pass through a fixed point of the basic line.*

12. Since the points of contact T_n , T_n' of the tangents from the variable point P_n of OP_n to the variable cubic Γ_n describe two straight lines (7), hence the variable line $T_n T_n'$ meets ABC in a fixed point (8). Thus: *The chords of contact of the pairs of tangents drawn to the cubics of the pencil Γ from the points which these curves respectively determine on a fixed line passing through the double point, meet the basic line in a fixed point.*

13. The chord of contact of the two tangents from A_n' (2) to Γ_n passes through the conjugate A of A_n' on Γ_n ,³ and meets the line $A_n' M_n M_n'$ (3) in the harmonic conjugate X' of the point $(OA, M_n M_n')$ with respect to the couple M_n , M_n' .⁴

The pairs of lines projecting from A the couples of corresponding points on Γ_n form an involution of rays, one of the double elements of which is the line AO joining A to the double point O . Since M_n , M_n' are a pair of corresponding points (3), the line AX' is the second double element of this involution of rays.

The line AX' has thus a definite geometric meaning relative to the point A on Γ_n . It is sometimes more convenient to speak of this line in connection with the point A alone, without having to mention the corresponding point of A . Such a line as AX' shall be referred to as "the double line" through A relative to Γ_n .

When A_n' varies with Γ_n (2) the harmonic pencil $O(M_n M_n' AX')$ has three rays OM_n , OM_n' (1), OA , fixed, hence the fourth ray OX' is also fixed. Consequently: *The double line, relative to a curve of the pencil Γ , through a basic point meets the chord of contact of the two tangents to this cubic from the same basic point, on a fixed line passing through the double point.*

It may readily be shown that the line OX' is identical with the locus OD_n of the tangentials of A (4).

14. Let K be the point of intersection of the line $P_n R_n$ (8) with ABC , and W_n the third point of intersection of $P_n R_n$ with Γ_n . Let P_s , R_s , W_s be the points of intersection of the lines OP_n , OR_n , OW_n with any other cubic, say Γ_s , of the pencil Γ . The lines $P_n R_n$, $P_s R_s$ intersect in K , by virtue of prop. (8), i.e., P_s , R_s , K are collinear.

Again, the lines $P_n W_n$, $P_s W_s$, intersect on ABC , by virtue of the same prop. (8). But $P_n W_n \equiv P_n R_n$ by construction, hence $P_s W_s$ passes through K , i.e., P_s , W_s , K are collinear, consequently P_s , R_s , W_s are collinear. Thus: (a) *If three lines passing through the double point meet one of the cubics of the pencil Γ in three collinear points, they meet every cubic of the pencil in three collinear points.* (b) *The bases of these triads of points concur on the basic line of the pencil.*

15. The lines $P_n A$, $P_n A_n'$ projecting from the point P_n (5) the couple of corresponding points A , A_n' (2) on Γ_n meet Γ_n again in a pair of corresponding points.⁶ The line $P_n A$ meets Γ_n again in Q_n (6), hence $P_n A_n'$ meets Γ_n again in the corresponding point Q_n' of Q_n on Γ_n . Thus the corresponding points P_n' , Q_n' of P_n , Q_n on Γ_n are collinear with A , and therefore the lines OP_n' , OQ_n' , are conjugate in the involution (I) (1).

The lines AP_n , AP_n' , projecting from A the pair of corresponding points P_n , P_n' are harmonically separated by the double lines AO , AX' of the involution projecting from A the pairs of corresponding points on Γ_n (13). The line OX' cuts

⁶ de Jonquières, loc. cit., p. 239, prop. XV.

the harmonic pencil $A(OX'P_nQ_n)$ in four harmonic points O, X', H_s, H_s' . The corresponding points P_n', Q_n' of the points P_n, Q_n are thus the intersection of AH_s' with the harmonic conjugates OP_n', OQ_n' of OP_n, OQ_n with respect to OT_1, OT_2 .

If the secant $s \equiv AP_nQ_n$ is maintained fixed, while Γ_n varies, the pair of lines OP_n, OQ_n will describe the involution (I) of which the pair of points P_n, Q_n will describe the section by s . The lines OP_n', OQ_n' will describe the same involution (I). The line OX' will remain fixed (13), the points X' and $U(8)$ lying on the line $A_1A_n'M_nM_n'$ (2, 3). Of the four harmonic points O, X', H_s, H_s' the points O, H_s are fixed, hence the two others H_s', X' will describe on the fixed line OX' two superposed projective ranges the double elements of which are the points O and H_s . Thus:

$$O(P_n'Q_n'; \dots) \propto O(P_nQ_n; \dots) \propto (P_nQ_n; \dots) \propto (U \dots) \\ \propto A_1(U \dots) \propto (X' \dots) \propto (H_s' \dots) \propto A(H_s' \dots).$$

This construction establishes a one-to-two projective correspondence between the pencil of lines $A(H_s' \dots)$ and the pairs of lines of the involution (I). Consequently⁷: *The corresponding points, on the cubics of the pencil Γ , of the points determined by this pencil on a fixed line passing through a basic point, lie on a nodal cubic having the same double point as the given pencil and passing through the basic point considered.*

When the lines OP_n, OQ_n coincide with the couple OB, OC of (I) the above construction shows that the harmonic conjugates of OB, OC with respect to the couple OT_1, OT_2 are the two tangents to this cubic at its double point O .

16. The lines OP_n, OQ_n (8) being harmonically separated by the double elements OM_n, OM_n' of (I), the points $U_p \equiv (OP_n, M_nM_n')$, $U_q \equiv (OQ_n, M_nM_n')$ are harmonically separated by M_n, M_n' . On the other hand M_n, M_n' being a couple of corresponding points on Γ_n (1), the lines P_nM_n, P_nM_n' are conjugate in the involution of lines projecting from P_n the couples of corresponding points on Γ_n , and since one of the double elements of this involution of rays is P_nO , therefore the second double element passes necessarily through the point U_q . Thus P_nU_q is the double line (13) of P_n relative to Γ_n . Similarly the line Q_nU_p is the double line relative to Γ_n through Q_n .

⁷ de Jonquières, loc. cit., pp. 170-171.

The four points P_n, Q_n, U_p, U_q , determine a complete quadrangle the diagonal points of which are $O \equiv (P_n U_p, Q_n U_q)$, $U \equiv (P_n Q_n, U_p U_q \equiv A_1 A_n')$, $U' \equiv (P_n U_q, Q_n U_p)$. Thus OU' is the harmonic conjugate of OU with respect to OP_n, OQ_n , hence OU' is identical with OA' (8), i.e., U' lies on the fixed line OA' . Let $E_s \equiv (OA', P_n Q_n)$. The harmonic pencil $U(E_s A_n' OU')$ determines on OA' the four harmonic points E_s, A_n', O, U' .

If the secant $s \equiv AP_n Q_n$ is kept fixed, while the cubic Γ_n varies, the pair of lines OP_n, OQ_n will describe the involution (I) of which the couple of points P_n, Q_n describe the section by the fixed line s . Of the four harmonic points E_s, A_n', O, U' the points O, E_s remain fixed, hence A_n', U' describe on the fixed line OA' two superposed projective ranges with the points O and E_s as the double elements of the projectivity. Thus:

$$(a) \quad (P_n Q_n; \dots) \propto (U \dots) \propto A_1 (U \dots) \propto (A_n' \dots) \propto (U' \dots).$$

When U' coincides with E_s , the point A_n' also coincides with E_s , hence U will also coincide with E_s , consequently one point of the couple of elements of the involution $P_n Q_n \dots$ coincides with E_s . Thus in the projective one-to-two correspondence established by (a) between the points of the lines OA' and s the point E_s is a united element, hence $U'P_n, U'Q_n$ envelop a conic tangent to s , the point of contact being the trace on s of the conjugate of OA' in (I), i.e., the line projecting from O the third point of intersection of Γ_n with AA_n' . Thus: *The double lines of the points determined by the cubics of the pencil Γ on a line passing through a basic point, envelop a conic tangent to the line considered.*

When the lines OP_n, OQ_n coincide with the lines OB, OC , the point A_n' and hence the point U' will coincide with O , consequently: *The conic is tangent to the lines projecting, from the double point, the other two basic points of Γ .*

17. The points in which a line l through the double point O meets the cubics of the pencil Γ have their tangentials on another line passing through O (6). Hence if l meets one of the cubics of Γ in a point which coincides with its tangential, the same is true about the intersection of l with any other cubic of Γ . Thus: *If a line through the double point meets one of the cubics of the pencil Γ in a point of inflexion, it meets all the cubics of the pencil in inflexional points.*

18. Let I_n' , I_n'' , I_n''' be the three points of inflection of Γ_n . The lines OI_n' , OI_n'' , OI_n''' meet every cubic of the pencil Γ in three collinear points (14a), and all these points are points of inflection (17). Consequently: *The points of inflection of all the cubics of the pencil Γ lie on three straight lines concurring in the double point.*

19. From (18) and (14b) it follows: *The inflectional lines of the cubics of the pencil Γ form a pencil having its vertex on the basic line.*

It may be added that this vertex is the harmonic conjugate, with respect to the couple A , A_1 , of the trace on ABC of the locus of the tangentials of A (4).

UNIVERSITY OF OKLAHOMA,
October, 1919.

DEFINITION AND ILLUSTRATIONS OF NEW ARITHMETICAL GROUP INVARIANTS.

BY PROFESSOR E. T. BELL.

1. ARITHMETICAL instances of groups are still sufficiently uncommon to make any new occurrence a matter of interest. Many significant group concepts have, of course, been implicit in arithmetic since at least the times of Euler and Gauss, notably in the theories of power residues, the automorphisms of binary quadratic forms, and principal genera. More recently Miller has directly applied groups to quadratic residues and other topics.

This note contains the definition and a few illustrations, shorn of algebraic and other details, of certain group invariant relations for arbitrary integers, which are believed to be fundamentally distinct from previous group phenomena in arithmetic. These relations are genuinely arithmetical in that they concern only integers, and they may legitimately be called group relations because they exist only in reference to groups.

The object of this note is merely to call attention to these invariants by exhibiting a few of the simplest; and as several preliminary definitions are required—the subject being new—developments and less obvious examples may be left to another

occasion. Although the illustrations have purposely been chosen of a very simple nature, nevertheless they sufficiently illustrate the general character of all such results, and notes on their analytical origins have been included. We point out here, however, that the elliptic, abelian, and allied theta functions offer inexhaustible mines of such invariants for systems of any number of quadratic and bilinear forms and for groups of any degree, and that applications of these invariants to numerous arithmetical problems are immediately apparent. For example, it will be clear from this note that any relation between theta functions and their allied quotients, provided only that it involves the arguments and is not merely an identity between constants, gives rise to at least one singular monad invariant, and therefore to several dyads, triads, etc. Regarding applications, the possible fruitfulness of the invariants will be evident on remarking that in any invariant the k -ads may be replaced by arbitrary functions of the same parity as the k -ads and belonging to the same group; so that each k -ad relation implies an infinity of specialized arithmetical theorems. Last, the whole method is obviously applicable to functions other than the thetas as a basis, provided they are expandible in Fourier series. But so far the most interesting applications have arisen by way of the theta functions.

2. The sets of independent variables, in which order is essential, $(x_1, x_2, \dots, x_a)$, $(-x_1, -x_2, \dots, -x_a)$, are denoted by ξ , $-\xi$ respectively. Consider $(r+s)$ such sets, all the variables being independent,

$$\xi_i \equiv (x_{i1}, x_{i2}, \dots, x_{ia_i}), \quad (i = 1, 2, \dots, r),$$

$$\eta_j \equiv (y_{j1}, y_{j2}, \dots, y_{jb_j}), \quad (j = 1, 2, \dots, s),$$

united in a symbol Z , in which order is essential,

$$Z \equiv (\xi_1, \xi_2, \dots, \xi_r | \eta_1, \eta_2, \dots, \eta_s),$$

which is called an $(r+s)$ -ad, of parity and order

$$(a_1, a_2, \dots, a_r | b_1, b_2, \dots, b_s), \quad \omega \equiv \sum_{i=1}^r a_i + \sum_{j=1}^s b_j$$

respectively, and which has the following properties: Z exists and is uniquely determined when all the variables take integral values ≥ 0 , remains equal to itself when any ξ is replaced

by its negative, changes sign for a like replacement of any η , and vanishes when all the variables in any η take zero values simultaneously. Likewise, by a few changes in the wording, we define the $(r+0)$ -ad Z' and the $(0+s)$ -ad Z'' ,

$$Z' \equiv (\xi_1, \xi_2, \dots, \xi_r), \quad Z'' \equiv (\eta_1, \eta_2, \dots, \eta_s),$$

of the obviously corresponding parities and orders $(a_1, a_2, \dots, a_r)$, Σa_i , etc. If $r+s=k$, $r+s>0$, we shall speak of Z, Z', Z'' simply as k -ads.

All the formal laws of algebraic addition and subtraction are to hold for k -ads; and two k -ads are equal when and only when it is possible, by legitimate changes of sign in accordance with the foregoing definitions, to reduce them to identically the same form, the signs prefixed to the whole k -ads being included. Thus $(2, 3|)$, $(-2, 3|)$, $(2, -3|)$, $(-2, -3|)$, are equal; $(2, 3|)$ and $(3, 2|)$ are unequal, as also are $((2, 3|)$ and $(2, 3|)$; $(|(2, 3)) = -(|(-2, -3))$; and

$$\begin{aligned} (1, (-2, 0, 3)|(-4, 0), 5) &= (1, (2, 0, -3)|(-4, 0), 5) \\ &= -(-1, (-2, 0, 3)|(4, 0), 5) = (1, (-2, 0, 3)|(4, 0), -5). \end{aligned}$$

By the variables of a k -ad we shall mean the variables within the sets ξ, η , and not the ξ, η which will be called the sets of the k -ad.* If each set contains but one letter, the sets and the variables coincide.

3. Let $1, s_1, s_2, \dots, s_t \equiv g$ denote a set of substitutions on some or all of the variables of a k -ad Z , and write $(s_i Z)$ for the result of applying s_i to Z . The s_i fall into two categories: (A), substitutions which either permute the variables of Z within the sets, interchanging no pair of variables from different sets, or permute the sets ξ among themselves and the sets η among themselves, interchanging no ξ with any η ; (B), all others. If all the substitutions of a group G belong to (A), we call G an A -group, ("for Z " being understood). In this note we consider only A -substitutions and A -groups, as the consequences are simpler than those arising from (B).

If g is any A -group on the variables of Z , we shall denote by Z^g any k -ad of the same parity as Z , and on the same variables, which belongs to g . As a simple example, the following sum is a $(1+0)$ -ad of parity $(4|)$ and order 4:

* This is emphasized because in one application of the theory the ξ, η behave as single variables, not as sets, and are then treated as the independent variables.

$$(x|y, (z, w)) + (x|w, (z, y)) + (y|z, (w, x)) + (y|x, (w, z)) \\ + (z|w, (x, y)) + (z|y, (x, w)) + (w|x, (y, z)) + (w|z, (y, x)),$$

which obviously is invariant for an octic group g on x, y, z, w and which therefore is a particular monad of that whole class any member of which is denoted by $((x, y, z, w)|)^g$. As another example, if λ, μ denote integers different from zero, and λ is even, μ odd, the function

$$x^\lambda(y^\mu - z^\mu) + y^\lambda(z^\mu - x^\mu) + z^\lambda(x^\mu - y^\mu)$$

belonging to $g \equiv 1, xyz, xzy$, is of parity $(|3)$ in (x, y, z) and is therefore a particular $(|(x, y, z))^\theta$ for this g . Last, recalling a common notation, we denote by $[\alpha\beta\gamma\delta]$, $[\alpha\beta\gamma\delta]'$ the products of any four elliptic theta functions $\vartheta_a(x)$, etc., in which the variables are, for the order in which the functions are written, x, y, z, w and x', y', z', w' respectively, where

$$2s \equiv x + y + z + w,$$

$$x' = s - x, \quad y' = s - y, \quad z' = s - z, \quad w' = s - w;$$

and we have either of the equal sums

$$[0123] + [1032] + [2301] + [3210],$$

$$[0123]' + [1032]' + [2301]' + [3210]'$$

a particular $(|(x, y, z, w))^\theta$ for g the four group 1, $xy \cdot zw$, $xz \cdot yw$, $xw \cdot yz$. We interpret this example arithmetically later. Although all of these are for monads, it is not difficult to devise similar examples for dyads, triads, etc. We pass over these, since from invariant relations concerning monads may be inferred others involving dyads, etc., as pointed out in § 12.

4. Consider a k -ad Z whose variables are $z_1, z_2, \dots, z_\omega$, and an A -group $g \equiv 1, s_1, s_2, \dots, s_r$ on some (or all) of these variables, and let

$$S: (z_1, z_2, \dots, z_\omega) \equiv (z_1^{(i)}, z_2^{(i)}, \dots, z_\omega^{(i)}), (i = 1, \dots, t)$$

denote a system of t integral values of the set $(z_1, z_2, \dots, z_\omega)$; viz., each $z_j^{(i)}$ is an integer ≥ 0 , and in the typical identity between sets just written, $z_j = z_j^{(i)}$ ($j = 1, 2, \dots, \omega$). Replace z_j by $z_j^{(i)}$ in Z , ($j = 1, 2, \dots, \omega$), multiply* the result

* From § 2 the meaning of cZ , where c is an integer, is $Z + Z + \dots + Z$ (c terms) if $c > 0$; it is $-Z - Z - \dots - Z$ if $c < 0$; and $cZ = 0$ when $c = 0$.

by an integer c_i , form the sum of all such products for $i = 1, 2, \dots, t$, and denote the result by $f_s cZ$, or by fcZ where S is understood. In the same way are defined $fc(s_i Z)$, fcZ^g , starting from $(s_i Z)$ and Z^g respectively. In these sums the substitutions s_i and those of g must obviously be applied before the variables of Z are replaced by the integers of S .

5. In all that follows there is a radical distinction between special and general sums of the sort just defined. In a special sum the integers $z_j^{(i)}$ are constants, $0, -2, 5, \dots$, etc.; in a general sum they are variables. Through several of the $z_j^{(i)}$ having equal absolute values we frequently find $fcZ = 0$ for Z special, but $fcZ \neq 0$ for Z general; and so in the other cases. Henceforth all f sums are general.

6. Bearing in mind that the f are general, we readily see that relations

$$fcZ = 0, \quad fc(s_i Z) = 0, \quad fcZ^g = 0,$$

must be mere trivial identities until the integers of S are in some way restricted. Hence we next impose conditions upon S , emphasizing again, however, that within the restrictions the integers of S are variables. For example let (x_i, y_i, z_i, w_i) , $(i = 1, \dots, h)$ represent the totality of sets of four integers > 0 , the sum of whose squares is equal to an odd integer m , and choose

$$S: (z_1, z_2, z_3, z_4) \equiv (x_i, y_i, z_i, w_i) \quad (i = 1, \dots, h).$$

Then in this case our sums are to refer to S for m an arbitrary odd positive integer; and we shall seek for this S a relation such as $fc((z_1, z_2, z_3, z_4) |)^g = 0$, valid for any odd positive integer whatever.

7. It follows at once from the definitions that if g is an A -group, and s_i a substitution of (A) , then the first of the f relations in § 6, wherein S is now restricted, implies each of the others. For Z^g is a special case of Z ; and $fc(s_i Z) = 0$ is $fcZ = 0$ with a changed notation. E.g., if $Z \equiv ((z_1, z_2, z_3, z_4) |)$ as above, and $s_1 = z_1 z_2 \cdot z_3 z_4$, $fc((z_1, z_2, z_3, z_4) |) = 0$ becomes, on applying s_1 , $fc((z_3, z_4, z_1, z_2) |) = 0$, which abstractly is the same thing; and similarly for A -substitutions, which interchange sets. Clearly the like does not hold for B -substitutions.

Suppose now that $fcZ = 0$. Then this implies $fcZ^g = 0$, as remarked, and this is an invariant relation for the A -group

g. We shall call this an improper invariant, it being included as a very special case in another relation, fcZ , between k -ads on the same variables. We proceed to define proper invariants of S for g , stating the definition for the general case. These express essential invariant properties of S ; the improper, considered as invariants, are trivial, the relations from which they follow being more general.

8. Let F_i ($i = 1, \dots, k$) denote a system of k algebraic forms, not necessarily all of the same orders and degrees, with integral coefficients, and consider all the representations of a fixed arbitrary integer n in the system Σ ,

$$\Sigma: \quad F_i = n \quad (i = 1, \dots, k).$$

Consider first only one of the equations in Σ ; say it is $F(w_1, w_2, \dots, w_h) = n$, and suppose it to have precisely t distinct solutions.* Let

$$(w_i) \equiv (w_1^{(i)}, w_2^{(i)}, \dots, w_h^{(i)}), \quad (i = 1, 2, \dots, t),$$

be these solutions. From the integers in one of them, (w_i) , construct ω linear homogeneous functions λ with constant rational coefficients, some of which may be zeros,

$$\lambda_p^{(i)} \equiv a_{p1}w_1^{(i)} + a_{p2}w_2^{(i)} + \dots + a_{ph}w_h^{(i)}, \quad (p = 1, 2, \dots, \omega).$$

With the notation at the beginning of § 4 choose

$$(z_1^{(i)}, z_2^{(i)}, \dots, z_\omega^{(i)}) \equiv (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_\omega^{(i)}), \quad (i = 1, 2, \dots, t);$$

make this substitution in fcZ , $fc(s_iZ)$, fcZ^g as there (§ 4) defined, and denote the results by

$$fc\lambda, \quad fc(s_i\lambda), \quad fc\lambda^g.$$

Returning to the system Σ , we proceed similarly with all of its equations, forming λ 's for each, the coefficients in the λ 's for the several forms F_j not necessarily being the same, and getting finally the sums

$$f_j c^{(j)} \lambda, \quad f_j c^{(j)} (s_i \lambda), \quad f_j c^{(j)} \lambda^g, \quad (j = 1, \dots, k),$$

where the suffix j signifies that the sums belong to the particular equation $F_j = n$, and j in $c^{(j)}$ distinguishes the constants in the several f sums. Next sum each of these for

* It is immaterial whether t is finite or infinite. In this note we illustrate only invariants for Σ 's having a finite number of solutions.

$j = 1$ to k , that is, over the entire system Σ , and write the results

$$\sum_{j=1}^k f_j c^{(j)} \lambda, \quad \sum_{j=1}^k f_j c^{(j)} (s_i \lambda), \quad \sum_{j=1}^k f_j c^{(j)} \lambda^g, \\ \equiv f_{\Sigma} c \Lambda, \quad f_{\Sigma} c (s_i \Lambda), \quad f_{\Sigma} c \Lambda^g,$$

respectively, or the same with the omission of Σ where it is understood.

Last, let $g \equiv 1, s_1, \dots, s_r$ be a group whose substitutions are unrestricted with respect to (A) , (B) on some or all of the variables of the Z from which Λ is derived as above. Then, if it be possible to choose the coefficients of the λ 's so that $f c(s \Lambda) \neq 0$ for at least one* substitution s , which may be the identity, of g , while $f c \Lambda^g = 0$, the latter relation is defined to be a proper g -invariant relation, or simply g -invariant, of Σ .

9. It can be shown without difficulty that the first of the following implies the second,

$$f c \Lambda + \sum_{i=1}^r f c (s_i \Lambda) = 0, \quad f c \Lambda^g = 0,$$

all the notation being as in § 8. Indeed this is an immediate consequence of the definitions, and we may omit a formal proof. Again if γ is any subgroup of g , all of whose substitutions not in γ belong to (A) , it is easily seen that of the following the second is implied by the first,

$$f c \Lambda^\gamma = 0, \quad f c \Lambda^g = 0.$$

The like does not hold if any substitution of g not in γ belongs to (B) . We shall say that the g -invariant is contained in the γ -invariant; and writing these for the moment I_g, I_γ respectively, symbolize the inclusion thus, $I_g > I_\gamma$. If now $I_\gamma > I_\delta, I_\delta > I_\beta, \dots, I_\alpha > I_\epsilon$, we have

$$I_g > I_\gamma > I_\delta > I_\beta > \dots > I_\alpha > I_\epsilon,$$

and clearly any such series of inclusions contains only a finite number of terms. If I_ϵ is the last term, viz., if there is no group ζ different from ϵ such that $I_\epsilon > I_\zeta$, we call I_ϵ the reduced invariant of $I_\alpha, I_\beta, \dots, I_g$. Henceforth invariant means reduced invariant. When ϵ is the identity, $I_\epsilon \equiv I_1$ is called a unit.

* If g is an A -group this implies $f c(s \Lambda) \neq 0$ for every s of g .

10. With the notation of § 8, let

$$\sum_{j=1}^k f_j c^{(j)} \lambda \equiv f_z c \Lambda \equiv f c \Lambda = 0$$

be a unit; and let $1, \sigma_1, \sigma_2, \dots, \sigma_{k-1}$ denote a set of substitutions belonging to (A) on the variables of the Z from which this I_1 is derived. Write $f c \Lambda = 0$ in full,

$$f_1 c^{(1)} \lambda + f_2 c^{(2)} \lambda + \dots + f_k c^{(k)} \lambda = 0,$$

and put

$$I \equiv f_1 c^{(1)} \lambda + f_2 c^{(2)} (\sigma_1 \lambda) + \dots + f_k c^{(k)} (\sigma_{k-1} \lambda).$$

We cannot infer $I = 0$, that is, $I = 0$ is not a unit invariant; but if the σ are substitutions of an A -group g which are not all contained in a group of order lower than that of g , then clearly $I = 0$ is a g -invariant (proper and reduced) of Σ . We shall call $I = 0$ a singular g -invariant of Σ . Several of the F_j in Σ may be identical. For such Σ the unit invariant $f c \Lambda = 0$ appears in the form

$$f_1 c^{(1)} \lambda + f_2 c^{(2)} (s_1 \lambda) + \dots + f_k c^{(k)} (s_{k-1} \lambda),$$

where the s_j are substitutions on the variables of Z . As before, we get a singular g -invariant $J = 0$, where

$$J \equiv f_1 c^{(1)} \lambda + f_2 c^{(2)} (s_1 \sigma_1 \lambda) + \dots + f_k c^{(k)} (s_{k-1} \sigma_{k-1} \lambda),$$

on choosing $s_1 \sigma_1, s_2 \sigma_2, \dots, s_{k-1} \sigma_{k-1}$ so that they do not all occur in a group whose order is lower than that of g .

11. Taking any g -invariant for Σ as a primitive, we deduce from it another g -invariant called the derivative of the primitive, as follows. Let Z be the k -ad from which, as in § 8, $f c \Lambda^o = 0$ is constructed, and let $\xi_1, \xi_2, \dots, \xi_k$ be the sets of Z . Call $f(\xi_i)$ an even or an odd function of ξ_i according as $f(\xi_i) = f(-\xi_i)$ or $f(\xi_i) = -f(-\xi_i)$. In each set to the left of the bar in Z replace each variable by an arbitrary odd or even function of the set in which it occurs, and in each set to the right of the bar in Z replace each variable by an arbitrary odd function of the set in which it occurs, getting thus a new k -ad Z' which is of the same parity in the variables of Z as is Z . Starting from Z' construct the sum $f c \Lambda'^o$ in precisely the same way that $f c \Lambda^o$ is constructed, viz., by substituting for the z -variables in Z the λ 's determined by Σ , and summing.

Then clearly the first of the following implies the second, which is called the derivative of the first,

$$f c \Delta^g = 0, \quad f c \Delta'^g = 0.$$

Clearly a primitive is neither more nor less general than its derivative. Nevertheless a derivative frequently suggests applications to specific problems more readily than does the primitive.

12. We write $f((x, y)|)^g = 0$, $f((x, y, z)|)^g = 0$, etc., understanding that the f sums refer to all (x, y) , (x, y, z) , $\dots$, the $x, y, z, \dots$ in any case being λ 's determined as in § 8 for a system Σ .

The number of implications increasing rapidly with the order of the k -ads, we shall write out the complete sets for orders 2, 3 only.*

Consider now the following tables.

- (I) $((x, y)|): (x, y|), (|x, y).$
- (II) $(|(x, y)): (x|y), (y|x).$
- (III) $((x, y, z)|): (x, (y, z)|), (|x, (y, z));$
 $(y, (z, x)|), (|y, (z, x));$
 $(z, (x, y)|), (|z, (x, y));$
 $(x, y, z|), (x|y, z), (y|z, x), (z|x, y).$
- (IV) $(|(x, y, z)): (x|(y, z)), ((y, z)|x);$
 $(y|(z, x)), ((z, x)|y);$
 $(z|(x, y)), ((x, y)|z);$
 $(|x, y, z), (y, z|x), (z, x|y), (x, y|z).$

The meaning of these will be seen from the first row of (III). It is: $f((x, y, z)|)^g = 0$ implies $f(x, (y, z)|)^g = 0$ and $f(|x, (y, z))^g = 0$; and inversely the last two together, but neither singly, imply the first. Similarly for the pairs and

* A systematic rule for deducing all the implications for order k will be evident on reading § 9, p. 319, of a paper in this BULLETIN, vol. 25, 1919. A proof of the general theorem for units is given in the Introduction to Part 1 of "Arithmetical paraphrases," presented to the Society, Oct., 1918, and the result for $I_g, g \neq 1$, follows at once from this. § 13 is also proved (in much more general form) in the same place.

set of four in the remaining rows of (III); likewise for (I)–(IV). For a monad invariant of parity $(4|)$, $f((x, y, z, w)|)^{\sigma} = 0$ implies $f(x, (y, z, w)|)^{\sigma} = 0$ and $f(|x, (y, z, w))^{\sigma} = 0$; these together imply the original; and from (I)–(IV) the complete set of implications may be written out for order 4, parity $(4|)$. For $f(|(x, y, z, w))^{\sigma} = 0$ the first pair of implications is $f(x|(y, z, w))^{\sigma} = 0$, $f((y, z, w)|x)^{\sigma} = 0$; and the procedure for monads of parities $(k|)$, $(|k)$ is evident. By the notation

$$ff((x, y, z)|)^{\sigma} = 0$$

we shall mean the complete set of eleven invariants written down from (III) by replacing $((x, y, z)|)$ in $f((x, y, z)|)^{\sigma} = 0$ by each of the eleven dyads and triads in (III), and similarly for order k .

13. It remains to indicate the connection of these invariants with the theta and allied functions. From any identity between such functions in the variables $\alpha, \beta, \dots, \gamma$, we deduce trigonometric identities in $\alpha, \beta, \dots, \gamma$, of which the following is one type,

$$\Sigma c \cos(\alpha x + y\beta + \dots + z\gamma) = 0,$$

the summation referring to the constants c and the integers $x, y, \dots, z$, the latter being the λ 's of § 8. It can be proved without difficulty, but at some length (cf. § 12, footnote) that this trigonometric identity implies $fc((x, y, \dots, z)|) = 0$. If the identity involves sines instead of cosines, we get $f(|(x, y, \dots, z)) = 0$. Groups enter either ab initio in the irreducible form of the theta identity, giving rise to trigonometric identities that are invariants under substitutions on the $\alpha, \beta, \dots, \gamma$, or in the derivation of singular from unit invariants as already indicated.

The following examples, the last of which is numerical, illustrate the principal definitions.

14. (i) As a first example let Σ (§ 8) be the single equation

$$2m = l_1^2 + l_2^2 + m_3^2 + m_4^2,$$

wherein m is odd, l_1, l_2 even ≥ 0 , m_3, m_4 odd ≥ 0 ; write $(|(x, y, z, w))^{\sigma} \equiv \{x, y, z, w\}$ where g is the four group in § 3; and

$$2\sigma \equiv l_1 + l_2 + m_3 + m_4, \quad \lambda_i \equiv \sigma - l_i, \quad \mu_i \equiv \sigma - \mu_i.$$

Then the theta identity in § 3 leads to a trigonometric identity (on equating coefficients of q^{2m}) which, as indicated in § 13, paraphrases by means of §§ 13, 12 into the set of invariants

$$ff[\{l_1, m_3, m_4, l_2\} - \{\lambda_1, \mu_3, \mu_4, \lambda_2\}] = 0.$$

(ii) Similarly, from one of Kronecker's forms of the "equation of three terms," * for $g \equiv 1, yzw, ywz$, and $((x, y, z, w)|)^g \equiv \{x, y, z, w\}$ we get when Σ is the equation

$$4n = m_1^2 + m_2^2 + m_3^2 + m_4^2,$$

where m_1, m_2, m_3, m_4 are odd ≥ 0 , on writing $2\mu \equiv m_1 + m_2 + m_3 + m_4$, the invariants

$$ff(-1)^\mu \{m_1 + m_2, m_1 - m_2, m_3 + m_4, m_3 - m_4\} = 0.$$

(iii) From one of Briot and Bouquet's forms of the same equation,† we find for the same Σ, g , the invariants

$$ff(-1)^\alpha \{m_1 + m_3, m_2 + m_3, m_1 + m_2, m_3 + m_4, m_3 - m_4\} = 0,$$

where now $((u, v, y, z, w)|)^g \equiv \{u, v, y, z, w\}$; α is either μ or σ and these are

$$2\mu \equiv m_1 + m_2 + m_3 + m_4, \quad 2\sigma \equiv m_3 + m_4.$$

(iv) From the same source, for Σ as in (i), and $g, \{u, v, y, z, w\}$ as in (iii),

$$2\rho \equiv l_1 + l_2 + m_3 + m_4, \quad 2\sigma \equiv m_3 + m_4, \quad 2\nu + 1 \equiv l_1 + m_4;$$

$$ff(-1)^\beta \{l_1 + m_3, l_2 + m_3, l_1 + l_2, m_3 + m_4, m_3 - m_4\} = 0,$$

where β is ρ, σ or ν .

(v) From the same source, for Σ the system of three equations, m odd,

$$2m = u_1^2 + v_1^2 = x_2^2 + u_2^2 + v_2^2 = x_3^2 + y_3^2 + u_3^2 + v_3^2,$$

$$u_1, v_1, u_2, v_2, u_3, v_3 \text{ odd } \geq 0, \quad x_2, x_3, y_3 \text{ even } \geq 0, \text{ and } g, \{ \}$$

as in (iii), $2\mu_i = u_i + v_i$, we find

$$\begin{aligned} &ff[(-1)^{\mu_1}\{u_1, 0, u_1, v_1, -v_1\} \\ &+ (-1)^{\mu_2}\{y_3 + u_3, x_3 + y_3, x_3 + u_3, y_3 + v_3, y_3 - v_3\}] \\ &+ ff(-1)^{\mu_3}[\{u_2, x_2, x_2 + u_2, v_2, -v_2\} \\ &+ \{x_2 + u_2, x_2, u_2, x_2 + v_2, x_2\}] = 0. \end{aligned}$$

* *Crelle*, vol. 102, p. 262 (F).

† *Théorie des Fonct. ellip.* (éd. 2, 1875), p. 488.

(vi) The identity, with $\varphi_1(x, y) \equiv \vartheta_1' \vartheta_1(x + y) / \vartheta_0(x) \vartheta_0(y)$,

$$\vartheta_0(x) \vartheta_0(y) \vartheta_0(z) \varphi_1(x, y) = \vartheta_1' \vartheta_0(z) \vartheta_1(x + y),$$

on expanding $\varphi_1(x, y)$ in a Fourier series and paraphrasing as outlined in (i), gives invariants for another kind of Σ , viz.,

$$2m = x_1^2 + y_1^2 + z_1^2 + 2u_1v_1 = x_2^2 + u_2^2 + v_2^2,$$

where x_1, y_1, z_1, x_2 are even $\cong 0$, u_1, v_1, u_2, v_2 odd, $u_1, v_1 > 0$, $u_2, v_2 \geq 0$;

$$\iint [4(-1)^a \{x_1 + u_1, z_1, y_1 + v_1\} + (-1)^b \{u_2, x_2, u_2\}] = 0,$$

where $\{x, y, z\} \equiv (|(x, y, z)|)^g$, $g \equiv 1, xyz, xzy$, and

$$2\alpha \equiv x_1 + y_1 + z_1, \quad 2\beta \equiv x_2 + u_2 + v_2.$$

(vii) From (vi) we get, in the same notation except g , the following invariants for g the symmetric group on x, y, z ,

$$\iint [4(-1)^a \{x_1 + u_1, y_1 + v_1, z_1\} + (-1)^b \{u_2, u_2, x_2\}] = 0.$$

(viii) Putting, for the m 's odd ≥ 0 , the l 's even $\cong 0$,

$$\mu_1 = m_1 + m_2, \quad \mu_2 = m_3 + m_4, \quad \mu_3 = m_3 - m_4,$$

$$\lambda_1 = l_1 + l_2, \quad \lambda_2 = l_3 + l_4, \quad \lambda_3 = l_3 - l_4,$$

$$\mu_4 = m_1 + m_3, \quad \mu_5 = m_2 + m_3,$$

$$\lambda_4 = l_1 + l_3, \quad \lambda_5 = l_2 + l_3,$$

we get (from the same source as (iii)) singular invariants for the system

$$4n = m_1^2 + m_2^2 + m_3^2 + m_4^2 = l_1^2 + l_2^2 + l_3^2 + l_4^2,$$

$$2\mu \equiv m_1 + m_2 + m_3 + m_4,$$

on writing $\{x, y, z, u, v\} \equiv ((x, y, z, u, v)|)^g$, as follows:

$$\begin{aligned} \iint [(-1)^a \{\mu_3, \mu_2, \mu_1, \mu_4, \mu_5\} + \{\lambda_3, \lambda_4, \lambda_2, \lambda_1, \lambda_5\} \\ - \{\lambda_2, \lambda_3, \lambda_1, \lambda_4, \lambda_5\}] = 0, \end{aligned}$$

for g the octic group generated by xz, yu ; for g the tetrahedral group generated by $xyz, xy \cdot zu$, a precisely similar $\iint$ with the suffixes of the respective $\{ \}$, in this order, 23145, 13425, 23145; for g the octahedral group generated by xy, yzu , the like with suffixes 21345, 32415, 23145; and for g the icosahedral

group generated by $xy \cdot zu$, xzv , the like with suffixes 21435, 21543, 23145.

(ix) As a numerical example, we take the invariant, from the same source as (vi), for $g = 1$, xy , and $\{x, y\} \equiv (|(x, y))^\theta$,

$$\iint \left[2(-1)^\lambda \{x_1 + z_1, y_1\} + (-1)^\mu \left\{ \frac{x_2 + y_2}{2} + z_2, z_2 \right\} \right] = 0;$$

$$\Sigma: 2m = 2x_1y_1 + 4z_1^2 = x_2y_2 + z_2^2,$$

m odd, x_1, y_1, x_2, y_2 odd > 0 , z_1 arbitrary ≥ 0 ; z_2 odd ≥ 0 ; $\lambda \equiv z_1$, $2\mu \equiv y_2 + z_2$. Let $m = 5$; then all the solutions of Σ are

$$(x_1, y_1, z_1) = (1, 5, 0), (5, 1, 0), (1, 3, \pm 1), (3, 1, \pm 1);$$

$$(x_2, y_2, z_2) = (1, 9, \pm 1), (9, 1, \pm 1), (1, 1, \pm 3), (3, 3, \pm 1).$$

Whence, substituting in the left of the invariant, and dropping the $\iint$, we get

$$\begin{aligned} & 2\{1, 5\} + \{5, 1\} - \{3, 3\} - \{5, 1\} - \{-1, 3\} - \{1, 1\} \\ & + [-\{-3, 1\} + \{1, 1\} - \{5, 1\} + \{-5, -1\} \\ & - \{-1, -1\} + \{3, -1\} + \{3, 3\} - \{-3, -3\}]. \end{aligned}$$

Since $\{x, y\}$ belongs to g , $\{x, y\} = \{y, x\}$; hence the sum reduces to

$$\begin{aligned} & [\{1, 5\} + \{-1, -5\}] - [\{3, 3\} + \{-3, -3\}] \\ & - [\{-1, 3\} + \{1, -3\}] - [\{1, 1\} + \{-1, -1\}]. \end{aligned}$$

By § 12 (II) this should vanish when $\{x, y\}$ is any one* of $(|(x, y))$, $(x|y)$, $(y|x)$ each belonging to g . Taking the first of these we have $\{-1, -5\} \equiv (|(-1, -5)) = -(|(1, 5))$, so that the first $[\] = 0$; similarly in this case for the others. Verifying for $(x|y)$, we have $(-1|-5) = -(1|5)$; hence the first $[\] = 0$; the third $= (-1|3) + (1|-3) = (1|3) - (1|3) = 0$; the fourth $= (1|1) + (-1|-1) = (1|1) - (1|1) = 0$. The verification for $(y|x)$ is similar.

UNIVERSITY OF WASHINGTON.

* The cases of $(x|y)$, $(y|x)$ are trivial in this instance, because it is readily seen that each vanishes under the given conditions for all values of x, y .

MATRICES AND DETERMINOIDS.

Matrices and Determinoids. By C. E. CULLIS. University of Calcutta Readership Lectures. Cambridge University Press, Vol. I, 1913, xii + 430; Vol. II, 1918, xxiv + 555 pp.

THIS treatise was designed to occupy two volumes of theory and one of applications to vector analysis and invariants. The growth of the manuscript however has brought about three volumes for the theory and one for applications. Consequently the present volume instead of closing the theory leaves it still incomplete. The third volume of theory is to include the theory of matrices with functions as coefficients, if we read the indications correctly.

The greater part of the first volume is devoted to the notion of determinoid and theorems connected with this. The matrix itself is studied mainly in connection with the notions of addition, subtraction, and multiplication. A study of the solution of matrix equations of the first degree, which includes systems of linear algebraic equations, also is included in this volume.

The second volume deals with compound matrices, the minors of a matrix, some properties of square matrices, rank of a matrix, transformations of a matrix, equations of the second degree, extravagances of matrices, paratomy and orthotomy of matrices, and three appendices.

As has happened in treatises from some Cambridge mathematicians, a great addition to the existing mathematical vocabulary is to be found in this treatise. Whether so many new terms are necessary of course remains to be seen in their developmental use. One's first impression is, however, that the matter is overdone. Some of them are descriptive enough to explain themselves to some extent, but others are manufactured for the occasion and only to be understood by reference to the text or a glossary. There is a complete and systematic notation throughout, which is highly desirable, and after one has accustomed himself to its method, it is quite intelligible, though successive abbreviation renders it more and more compact.

Definition of Matrix.—The author follows the usual custom and defines a matrix as an assemblage of m rows of n elements

each (where m and n are finite for the purposes of this treatise), the elements being numbers in the cases actually considered, and so far as one observes chosen from the real domain usually, but sometimes from the complex domain. The number of rows m is the horizontal order, and the number of columns n is the vertical order. The smaller of these two numbers if unequal, or either if equal, is the efficiency of the matrix, which is thus the order of the maximum determinants that may be formed from the elements of the matrix. Long rows, short rows, leading element, leading line or diagonal, leading vertical row, and leading position, are self-explanatory.

Regarding this definition the reviewer desires to make some comments which apply also to the definitions usually given of vector, tensor, etc. In the first place the term *assemblage of elements* almost invariably leaves out a highly essential phrase: in a certain prescribed order. For instance if the element 2 occurs in any of these entities it has a very different rôle when it is the first element, or the third element, or the second of the third set, etc. This is obvious. Hence the mere assignment of the values of the elements does not define uniquely the entity, whether matrix, vector, or similar entity. If one speaks of the vector (2, 3, 4) he tacitly implies certain notions: namely, what the position of the 2 signifies, the position of the 3, and the 4. The position is actually of more importance than the element in most problems. The definition of a vector as a triple of elements is not sufficient, nor is the definition of a matrix as an m -tuple of n -tuple sets. There seems to be then no valid reason why the qualitative element which designates the rôle of the numerical element should not be put in evidence. For instance, the vector above should be written $2\epsilon_1 + 3\epsilon_2 + 4\epsilon_3$, the $\epsilon_1, \epsilon_2, \epsilon_3$ being qualitative elements, hypernumbers of a unitary character. Likewise a matrix is expressible completely by the sum $\sum a_{ij} \lambda_{ij}$, where $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$. The hypernumbers λ indicate the position and the rôle of the coefficients. By thus placing such "assemblages" in the domain of linear algebras, where they properly belong, the whole treatment becomes simple and clear. Particularly the multiplication of matrices is set forth in a much more brilliant light. Further it should be noticed that a matrix of m long rows and n short rows is as much an assemblage of m vectors in an n -dimensional space, or of n vectors in an m -dimensional space, as it is of mn -ele-

ments. A matrix may be looked upon as a linear homogeneous substitution, which converts certain variables into others, or it may be considered as a linear vector operator which converts all vectors from the origin into others from the origin, or it may be considered as a dyadic, the sum of several dyads,—all useful definitions and yielding very important results. As a linear vector operator the particular elements entering the matrix are not of so much importance as other features of the matrix. Certain combinations of the elements are invariant, however, in various representations of the linear vector operator, or certain transformations of the matrix considered as an aggregate of elements. These combinations are of great importance. There are other matrices, which have been called covariants of the given matrix, which play a very important part in the theory, when we consider the matrix as an operator.

These considerations lead us to conclude that the mere assemblage of mn numerical values out of whose combinations various forms arise subject to a large variety of theorems is far from being the whole story.

Determinoids.—By determinoid the author means a sum of all the maximum determinants with specified signs that can be formed from the array of elements. Of course if the array is square there is but one such: the determinant of the matrix, as usually understood. If the matrix is rectangular, there must be assigned a rule for the addition of the maximum determinants, and many pages are devoted to the exposition of this rule in various forms, and to various modes of writing out the expansions of the determinoid.

Again the reviewer would remind the reader that out of the mn elements of a matrix an unlimited number of numerical combinations may be formed according to assigned rules, and the only question proper with regard to them is whether they are useful. Of all combinations those which are linear in the elements of each line or column would naturally be suggested as the most useful. Of these the products which are such as to have a number of factors equal to the order of efficiency of the matrix, would take priority. Products of elements chosen one from each of several different lines are called by the author derived products. If the efficiency is r and products of order r are formed in every possible manner from the elements of the matrix, so that no line is represented

twice in any one product, the sum of these products with arbitrary coefficients would be one of the useful combinations spoken of above. But the choice of the arbitrary coefficients would have to be governed again by some pragmatic principle. If they are all taken equal to ± 1 then the whole combination has a symmetry easily seen. If half are chosen properly positive and half properly negative, there is a skew symmetry. In each maximum minor we have such skew symmetry. The author was evidently aiming at some such result in his choice of coefficients for the derived products in forming his determinoid. But an examination of the theorems relating to determinoids, which he seems to think will be useful for the applications, makes it evident that to a large extent the coefficients of the maximum minors could just as well have been arbitrary and the theorems would still hold. We must then feel rather doubtful as to the utility of the determinoid when it is not a simple determinant.

To understand just what happens we may have recourse to the qualitative units. The elements of the row may be looked upon as defining a vector α_i . For simplicity we will suppose that the efficiency is m . Then if we set up a multiplication of these vectors by means of an alternating multiplication of the qualitative units $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ (as is often done in

Scott's Determinants, for instance), this will be an $\frac{n!}{m!(n-m)!}$ vector, of class m in a space of n dimensions, whose coordinates (coefficients) are the maximum minors of the matrix. As an alternating expression this product has certain properties and these are the properties which are to be found in the determinoid. For instance the addition of a long row to another is equivalent to adding one of the vectors to another, which will not affect the product. The determinoid itself is the scalar product of this alternating product and another vector of the same order, whose coefficients are ± 1 and -1 chosen according to the rules previously laid down. Any other vector of this order might in most cases as well have been chosen. We may reach the reciprocal matrix also with these alternating products, for if we construct a vector β_j of order 1, linear in the arbitrary vectors $\lambda_1, \lambda_2, \dots, \lambda_m$, any chosen set of m linearly independent vectors, the coefficient of λ_i being the scalar product of the alternating product of $\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_m$ and that of $\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_m$,

where j runs from 1 to m , and signs follow a determinant rule, then these vectors $\beta_1, \dots, \beta_j, \dots, \beta_m$, define the reciprocal matrix. In the case actually chosen the vector which is used to give the determinoid is also chosen as a sum of alternating products of m out of n vectors. Any arbitrary vector of the same order would do. In the case of a square matrix of course these all become multiples of a single unitary one, and the reciprocal of the square matrix which is not singular is therefore unique. Professor Cullis gets a unique reciprocal in the rectangular case only because of his special choice of the vector which gives the determinoid. By leaving this choice open for different problems he would have arrived at a much more flexible system. The product of a matrix and its reciprocal is of course the m th power of its determinoid. This would be true for any choice of the arbitrary vector.

Products of Matrices.—The elements of a product are formed by combining corresponding elements from the rows of the prefactor and the columns of the postfactor. These are the active rows of each respectively. The vertical rows of the prefactor and the horizontal rows of the postfactor are the passive rows. The passivity of either factor is the number of passive rows it contains. This is evidently a function of its place as prefactor or as postfactor. When the passivities are not equal they are made so by the adjunction of lines of zeros to the matrix with the smaller passivity. The product will then have the same number of horizontal rows as the prefactor, and vertical rows as the postfactor. The product of two vectors usually called their scalar product, can now be defined as the determinant (or determinoid equally in this case), of the product of two matrices, the first with one row, the second with one column. We may instead of inserting lines of zeros, strike out the redundant horizontal rows of the second or the redundant vertical rows of the first, and reach the same product. The latter method is preferable since it makes evident at once the rule for the cancellation of factors in a product which vanishes. This rule is stated for the most general case in these terms: The equation $AXB = 0$ leads to $X = 0$ as a necessary consequence, when and only when the ranks of A and B are equal to their passivities in the given product AXB . From this we have that $AX = AY$ leads to $X = Y$, as a necessary consequence, when and only when the rank of A equals its passivity in AX and in AY . And finally $AXB =$

AYB leads to $X = Y$, as a necessary consequence, when and only when the ranks of A and B equal their passivities in the two products. On the basis of these theorems we can proceed to solve linear equations in matrices or sets of linear equations in n variables, for these are reduced to linear matrix equations. For instance the system

$$\sum_{j=1}^n a_{ij}x_j = b_i, \quad (i = 1, \dots, m),$$

becomes the matrix equation

$$|a_{ij}||x_j| = |b_i|, \quad (i = 1, \dots, m; j = 1, \dots, n).$$

Compound Matrices.—These are defined and treated in the second volume. The notations are generalizations of those of the preceding volume. A compartite matrix is one whose elements all vanish except those belonging to a number of mutually complementary minors. By interchanging rows and columns it can always be brought to a form in which there are minors (usually rectangular) which form a diagonal set, all others being zero. This is called the standard form. The rank of a compartite matrix is the sum of the ranks of its component parts. The conjugate reciprocals of certain compound matrices are considered, the results being of use later. The primaries of the minor determinant Δ , of order r , in a matrix of orders m, n are the minors of order r which differ from Δ in only one row, the horizontal primaries differing in one horizontal row, the vertical in only one vertical row. The primary superdeterminants of a minor Δ of order r are all the minors of order r plus one which contain Δ as a minor. There are several theorems on the possible ranks of a matrix containing a given minor.

Relations between the Elements and the Minors.—In this chapter a quite considerable number of identities will be found, deduced for the most part by a skilful use of a matrix and its conjugate reciprocal. Many of them are identities well known in generalized vector analysis. In fact if we consider that the matrix of n short rows and m long rows is an aggregate of n vectors in an m -dimensional space, we can at once derive many of the formulas. Sylvester's identities satisfied by those primary superdeterminants of one determinant Δ which lie in another determinant D containing Δ are included in the

list. There is also a consideration of equivalent matrices, that is, matrices A, B such that $A = BC$ or $A = CB$, where C is an undegenerate matrix. These are used in the theorem that two systems of r unconnected linear equations on n variables which both have finite solutions are mutually equivalent when and only when the related matrices of coefficients are equivalent.

Square Matrices.—A large part of this long chapter is devoted to the properties of the square matrices whose elements are the minors of order r of a given matrix. We find such terms introduced as, co-joint matrices: each consisting of the minors which are complementary to the minors that constitute the other; corresponding and anti-corresponding minors: corresponding minors being formed in the same manner from corresponding elements of co-joint determinants, anti-corresponding minors being each the cofactor in one of two co-joint matrices to the minor corresponding to a given minor in the other. Several theorems are given relating to these matrices. Matrices that are formed by bordering a given matrix are studied. Reciprocal matrices give several theorems. Symmetric matrices and skew-symmetric matrices have some pages each. However, the whole subject of the characteristic equation of the matrix, the general equation, the scalar invariants, the related matrices, which the reviewer has called the chi functions of the matrix, are not even mentioned. Indeed after going over some 600 pages of the author's treatise, with the expectation that somewhere the matrix as an operator, not necessarily a linear substitution, but only an operator on other matrices, will be treated, one finds with disappointment that apparently this phase of the subject is not to be found in the author's scheme of development. The reduction of a matrix to its canonical form, the subject of elementary divisors, and all related topics are not mentioned. Perhaps the succeeding parts of the treatise will remedy this serious defect, but no hint that this will happen is given. The nearest approach seems to be in Chapter XVI, which deals with the equigradient transformations of a matrix with constants for elements.

Equigradient Transformations of a Matrix.—A transformation of this kind is equivalent to the transformation linearly of the variables in a bilinear form corresponding to the matrix. The greater part of the chapter is devoted to a consideration

of the various transformations that will reduce the matrix to a matrix consisting of square matrices down the main diagonal, other elements being zero. This corresponds of course to the reduction of the corresponding system of linear equations to a system which is reduced. In particular, symmetric matrices reduced to symmetric matrices are studied. The signants of the matrix are defined, and the matrix is called indefinite when there is both a positive signant and a negative signant, otherwise definite. The invariants of such transformations are not mentioned, nor is the significance of the transformation made clear.

Matrix Equations of the Second Degree.—The types considered are of course very special. We find equations of the unsymmetric forms

$$XY = AB, \quad XY = C.$$

All others considered are symmetric, such as, for instance, $X'X = I$, where X' is the transverse (conjugate) of X , and I is the identity matrix; $X'X = A'A$, $X'X = C$, $X'AX = C$. The case $X'X = I$ evidently gives the orthogonal, unitary matrices. This is solved in full in the sense that methods are given by which any number of special solutions may be constructed. An application is given to the rotations of a rigid body in three-dimensional space.

The Extravagances of Matrices.—The degeneracy of a matrix of rank r is the number by which its rank falls short of its efficiency, that is the number of long rows it contains. Or in other words, if a matrix is given by a set of vectors in a space of m dimensions (called a spacelet in the text) and its rank r is less than m , $m - r$ is the degeneracy of the matrix. The extravagance of an undegenerate matrix of rank r is the degeneracy of the self-transverse product of the matrix and its transverse. The extravagance is never negative nor more than the rank, nor more than the order minus the rank. An extravagant matrix is such that the sum of the squares of all its maximum minors is zero, that is to say, if we consider that the matrix is given by m vectors of an n -dimensional space, and form the alternant $A_m \alpha_1 \alpha_2 \dots \alpha_m$, the tensor or absolute magnitude of this vector is zero. A real undegenerate matrix evidently has zero extravagance. A completely extravagant matrix has extravagance equal to its rank. A matrix is plenarily extravagant when its extravagance is equal to the

difference of its orders. Matrices are mutually orthogonal when the product of one into the transverse of the other is zero, and mutually normal, when mutually orthogonal and also with ranks whose sum is n , the number of short columns in each. These terms lead to several theorems, and these to theorems as to spacelets. In a space of n dimensions these notions are connected with the absolute quadric.

Paratomy and Orthotomy.—A matrix of rank r may be considered to be an assemblage of vectors in a space of order r , called a spacelet, and if two spacelets have a spacelet in common as their intersection, then its rank is the mutual paratomy of the two. The mutual orthotomy is the degeneracy of the product of one matrix into the transverse of the other. There are several theorems involving these numbers, and these have evident applications to spaces of different dimensions.

The reviewer has endeavored to give as briefly as possible the main features of the treatise so far as it has been published, although the very large number of theorems and developments make this a difficult thing to accomplish. There remain to be added only a few general comments. The treatment is complete in the sense that it appeals to no other discipline for its methods or its proofs. Where vectors might have been used as operands they are always one-rowed matrices. One might question whether the same theorems might not be reached by other methods in less space and with much more direct connection with the field of applications. The reviewer believes that a judicious use of a generalized vector analysis (such as has been reported on in other places) would assist very much. However the author of a treatise must be permitted to develop it along his own lines and the really proper question is whether he succeeds in doing what he sets out to do, as he desires to do it. Of this there seems little doubt in considering the present treatise. If the remainder of the work is as full of useful theorems and applications, the completed treatise will remain for a long time a valuable place of reference. It is to be hoped that a complete bibliography will be added to the final volume, as well as a glossary. There are omissions which may be supplied in later volumes. For instance no work on matrices can be complete if it leaves out the consideration of the invariant regions, the projective regions, the shear regions of the matrix, the elementary divisors, the scalar and vector and

matrix invariants, the connection with dyadics and linear substitutions, the related triadics, polyadics, etc. This is a very extended field with numerous ramifications, and a complete treatise is in duty bound to consider these. On the side of determinants, which is the really major part of the treatise, one might insist on a consideration of the numerous special forms of determinants. The related matrices have interesting properties. Then finally the whole subject of groups of matrices leading to the field of linear associative algebra needs consideration, as well as the modern developments in the study of infinite matrices. There does not seem to be any indication that these are to be treated at all.

The treatment is quite detailed, with numerous numerical examples, rather loose in its development, and lacking in synthesis, so that the reader becomes bewildered with the multitudinous formulas and other details. A synopsis of it would be useful. There are some errors easily noticed

JAMES BYRNIE SHAW.

NOTES.

THE seventy-second meeting of the American association for the advancement of science was held at St. Louis, December 29 to January 3, under the presidency of Dr. SIMON FLEXNER. Professor O. D. KELLOGG was vice-president and Professor F. R. MOULTON secretary of Section A. The address of Professor G. D. BIRKHOFF, as retiring vice-president of Section A, on "Recent advances in dynamics," was delivered on December 30. This address was published in *Science* of January 16. Professor D. R. CURTISS was elected vice-president of Section A for the next two years. Among the societies meeting at St. Louis in affiliation with the Association were the Chicago and Southwestern Sections of the American Mathematical Society and the Missouri Section of the Mathematical Association of America.

THE fifth annual meeting of the Mathematical Association of America was held at Columbia University, New York City, on Thursday and Friday, January 1-2, immediately following the annual meeting of the American Mathematical Society.

The attendance included 107 members. The programme on Thursday included the following papers: "Mathematics for the physiologist and physician," by H. B. WILLIAMS; "The regular solids and the types of crystal symmetry," by P. L. SAUREL; "The mathematics of physical chemistry," by G. B. PEGRAM; "The mathematics of biometry," by L. J. REED, and "An experiment in conducting freshman mathematics courses," by F. B. WILEY; on Friday morning, a preliminary report of the National Committee on Mathematical Requirements, by J. W. YOUNG, and "Mathematics for students of physics," by LEIGH PAGE. At the business meeting on Friday, Professor D. E. SMITH was elected president, Professors HELEN A. MERRILL and E. J. WILCZYNSKI vice-presidents, and Professors R. D. CARMICHAEL, E. R. HEDRICK, H. E. SLAUGHT, and J. W. YOUNG members of the Council to serve until January, 1923. The Association held a joint dinner with the American Mathematical Society on Wednesday evening, December 31, at Students Hall, Barnard College.

According to its annual List of Members, issued November, 1919, the Mathematical Association of America then had a total membership of 1,187, including 87 institutional members. At the December meeting 75 new members were elected.

THE opening (January) number of volume 21 of the *Transactions of the American Mathematical Society* contains the following papers: "The strain of a gravitating sphere of variable density and elasticity," by L. M. HOSKINS; "The geometry of hermitian forms," by J. L. COOLIDGE; "Certain types of involutorial space transformations," by F. R. SHARPE and VIRGIL SNYDER.

THE concluding (October) number of volume 41 of the *American Journal of Mathematics* contains the following papers: "The ten nodes of the rational sextic and of the Cayley symmetroid," by A. B. COBLE; "Functions of matrices," by H. B. PHILLIPS; "On the Lüroth quartic curve," by FRANK MORLEY; "On the order of a restricted system of equations," by F. F. DECKER; "On the Lie-Riemann-Helmholtz-Hilbert problem of the foundations of geometry," by R. L. MOORE.

FIVE German publishers of scientific works, G. J. Göschen, J. Guttentag, G. Reimer, K. J. Trübner, and Veit, have united

in a single firm, the Vereinigung wissenschaftlicher Verleger, Walter de Gruyter und Compagnie, Berlin and Leipzig. The Berlin address is Genthinerstrasse 38, Berlin W. 10. This firm will continue the publication of the *Jahrbuch über die Fortschritte der Mathematik*.

PROFESSOR L. BERZOLARI, of the University of Pavia, and Colonel A. CROCCO, of the Aeronautic Institute of Rome, have been elected corresponding members of the R. Accademia dei Lincei.

PROFESSOR E. DANIELE, of the University of Catania, has been appointed professor of rational mechanics at the University of Modena.

SIR J. J. THOMSON has been elected foreign associate of the Paris Academy of Sciences, in place of the late Professor J. W. R. DEDEKIND.

MR. G. H. HARDY, fellow and mathematical lecturer of Trinity College, Cambridge, has been appointed to the Savilian professorship of geometry at Oxford University.

DR. J. PROUDMAN has been appointed professor of applied mathematics at the University of Liverpool.

PROFESSOR A. G. WEBSTER, of Clark University, has been elected honorary member of the Royal Institution of Great Britain.

PROFESSOR L. D. AMES has been made registrar and Mr. ELBERT ALLEN instructor in mathematics at the University of Missouri.

At Teachers College, Columbia University, Professor C. B. UPTON, of the department of mathematics, and secretary of the college since 1911, has been appointed provost.

MR. R. M. BARTON has been promoted to an assistant professorship of mathematics at the University of Minnesota.

DR. H. E. WOLFE has been appointed assistant professor of mathematics at Indiana University.

PROFESSOR L. S. SHIVELY, of the department of mathematics at Mount Morris College, has been made president of the college.

MR. T. W. JACKSON has been appointed head of the department of mathematics at Jamestown College.

MR. F. C. KENT has been promoted to an assistant professorship of mathematics at the Oregon Agricultural College.

MR. W. H. SHERK has been appointed professor of mathematics at the University of Buffalo.

DR. T. MCN. SIMPSON has returned from Y. M. C. A. service in France and has been appointed professor of mathematics at Randolph-Macon College.

MR. L. H. RICE, of Tufts College, has been appointed instructor in mathematics at the Massachusetts Institute of Technology.

MR. H. L. OLSON has been appointed instructor in mathematics at the University of Wisconsin.

MR. NORMAN ANNING has been appointed instructor in mathematics at the University of Maine.

THE following instructors in mathematics have been appointed at Purdue University: G. D. JAMES, T. S. RAIFORD, and R. S. UNDERWOOD.

PROFESSOR V. REINA, of the University of Rome, died November 9, 1919, at the age of fifty-three years.

PROFESSOR E. MILLOSEWICH, of the observatory of the Collegio Romano, died December 5, 1919, at the age of seventy years. He was a national member and the secretary of the R. Accademia dei Lincei.

PROFESSOR A. HURWITZ, of the Polytechnicum of Zurich, died November 18, 1919, at the age of seventy years.

PROFESSOR L. G. WELD, director of the Pullman school of manual training, died November 28, 1919. Professor Weld had been a member of the American Mathematical Society since 1892.

PRESIDENT R. C. MACLAURIN, of the Massachusetts Institute of Technology, died January 15, 1920. Dr. Maclaurin had been a member of the American Mathematical Society since 1908.

BOOK CATALOGUES: Macmillan Company, catalogue of 138 books on mathematics and astronomy.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- BOULIGAND (G.). Cours de géométrie analytique. Paris, Vuibert, 1919. 8vo. 421 pp. Fr. 10.00
- CLAPIER (F. C.). Sur les surfaces minima ou élassoides. (Thèse, Paris.) Paris, Gauthier-Villars, 1919. 4to. 63 pp. Fr. 3.00
- FIEDLER (W.). See SALMON (G.).
- GALEMBERT (—, DE). Nouvelles méthodes de résolution des équations du 3e degré. Paris, Vuibert, 1919. 8vo. 22 pp.
- KIEPERT (L.). Grundriss der Differential- und Integralrechnung. 1ter Band: Differentialrechnung. 13te Auflage. 1918. M. 20.00
- MONTESUS DE BALLORE (R. DE). Introduction à la théorie des courbes gauches algébriques. Paris, Croville-Morant, 1918. 4to. 112 pp.
- RUNGE (C.). Vektoranalysis. 1ter Band: Die Vektoranalysis des dreidimensionalen Raumes. Leipzig, Hirzel, 1919. 8 + 195 pp. Geb. M. 13.00
- SALMON (G.). Analytische Geometrie des Raumes. Deutsch von W. Fiedler. 2ter Teil: Kurven im Raum. 3te Auflage. 1916. M. 24.00
- SCHIEFFERS (G.). Lehrbuch der Mathematik. Eine Einführung in die Differential- und Integralrechnung und in die analytische Geometrie. 3te Auflage. 1916. M. 29.90
- THOMAS (R. G.). Applied calculus. Principles and applications; essentials for students and engineers. New York, Van Nostrand, 1919. 12mo. 500 pp. \$3.00
- TURC (A.). Introduction élémentaire à la géométrie Lobatschewskienne. Ouvrage posthume. Genève, Kündig, 1919. 8vo. 170 pp. Fr. 3.50

II. ELEMENTARY MATHEMATICS.

ALLEN (F.). See TAYLOR (E. H.).

BEHRENDSEN (O.) und GÖTTING (E.). Lehrbuch der Mathematik nach modernen Grundsätzen. Ausgabe A, für Gymnasien. Unterstufe. 3te Auflage. Leipzig, Teubner, 1917. 8 + 268 pp. Geb. M. 3.20

BUCK (R. C.). A manual of trigonometry. London, Griffin, 1919. 12mo. 8 + 133 pp. 3s. 6d.

DONADT (A.). See LÜBSEN (H. B.).

FRATTINI (G.). Lezioni di algebra, geometria e trigonometria piana e sferica, con molti esempi sull'intero programma del secondo biennio degl'istituti tecnici. Volume 1, per la terza classe. 4a edizione. Torino, G. B. Paravia (Roma, tip. Nazionale, Bertero), 1919. 8vo. 236 pp. L. 6.00

GENTLEMAN (F. W.). See VOSBURGH (W. L.).

GIULIANI (G.). Elementi di trigonometria piana, ad uso dei licei e degli istituti tecnici. 2a edizione. Torino, G. Chiantore, 1919. 8vo. 47 pp. L. 2.50

GÖTTING (E.). See BEHRENDSEN (O.).

HAMMER (E.). Lehr- und Handbuch der ebenen und sphärischen Trigonometrie. 4te Auflage. 1916. M. 19.00

HJELMSLEV (J.). Geometriske Eksperimenter. København, Gjellerup, 1919.

JORDAN (W.). Logarithmisch-trigonometrische Tafeln für neue (zentsimale) Teilung mit 6 Dezimalstellen. 2te Auflage. 1916. M. 14.40

LESSER (O.). See SCHWAB (K.).

LÜBSEN (H. B.). Ausführliches Lehrbuch der Elementargeometrie. Zum Schul- und Selbstunterricht mit Rücksicht auf die Zwecke des praktischen Lebens. In 2 Teilen. Neubearbeitet von A. Donadt, mit einer Anleitung zum perspektiven Zeichnen. 31te Auflage. 1918. M. 6.40

MARTINI (Z. A.). Algebra complementare. 3a edizione. (Biblioteca degli Studenti.) Livorno, R. Giusti, 1919. 16mo. 12 + 143 pp. L. 1.80

MORVAN (J.). Le barème pratique. Méthode complète de calculs rapides. Paris, Dunod et Pinat, 1919. 8vo. 141 pp. Fr. 7.20

MÜLLER (C. H.). See SCHWAB (K.).

NASSÒ (M.). Aritmetica generale ed algebra, ad uso dei licei, con copiose note storiche e molti consigli pratici per indirizzare l'alunno alla risoluzione degli esercizi. 10a edizione. Torino, libr. ed. Internazionale, 1919. 8vo. 530 pp. L. 4.60

PAGLIANO (C.). Sunti di algebra per il primo biennio degli istituti tecnici. 2a edizione. (Enciclopedia Scolastica.) Rocca S. Casciano, L. Capelli, 1919. 16mo. 98 pp. L. 0.60

SCHMIDT (H.). Auflösungen zu den Prüfungsfragen aus der Mathematik und Naturlehre. 2ter Teil. Leipzig, Haase, 1919. Geb. M. 4.20

- SCHWAB (K.) und LESSER (O.). Mathematisches Unterrichtswerk für höhere Lehranstalten. Lehr- und Uebungsbuch der Geometrie. 2ter Teil. Ausgabe B, für die Oberstufe der Gymnasien. Besorgt von C. H. Müller. 3te Auflage. Leipzig, Teubner, 1916. 242 pp. Geb. M. 3.00
- TAYLOR (E. H.) and ALLEN (F.). Junior high school mathematics. 2d book. New York, Holt, 1919. 9 + 251 pp.
- THAER (A.). Sammlung mathematischer Formeln für höhere Lehranstalten. Breslau, Hirt, 1919. M. 0.60
- VOSBURGH (W. L.) and GENTLEMAN (F. W.). Junior high school mathematics, third course. New York, Macmillan, 1919. 9 + 295 pp.

III. APPLIED MATHEMATICS.

- ABENDROTH (A.). Die Ausgleichungspraxis in der Landesvermessung. Eine Zusammenstellung der wichtigsten Ausgleichungsaufgaben bei Landestriangulierungen. 1916. M. 16.50
- AUBERT (J.). La probabilité dans les tirs de guerre. Paris, Gauthier-Villars, 1919. 8vo. 8 + 132 pp. Fr. 9.00
- AUERBACH (F.). Das Wesen der Materie. Leipzig, Durr, 1918.
- DOEHLEMANN (K.). Grundzüge der Perspektive nebst Anwendungen. 2te Auflage. Leipzig, Teubner, 1919. Geb. M. 1.90
- FLEURY (C.). Précis de technologie mécanique. Paris, Dunod et Pinat, 1919. 8vo. 545 pp. Fr. 27.00
- GANSBERG (F.). Der Flugzeugkompass und seine Handhabung. Kompasskompensieren. Kursabsetzen. Ein Handbuch für Flugzeugführer und Beobachter. 2te Auflage. 1917. M. 2.30
- GRAETZ (L.). Die Atomtheorie in ihrer neuesten Entwicklung. Stuttgart, J. Englehorn's Nachfolger, 1918.
- GRAF (E.). Technische Berechnungen für die Praxis des Maschinen- und Bautechnikern. Ein Handbuch über gelöste Beispiele aus der gesamten Mechanik, der Maschinen-, Holz- und Bautechnik, einschliesslich Eisenbeton- und Brückenbau. 2ter Abdruck. 1919. M. 16.00
- GROTRIAN (O.). Die Geometrie der Gleichstrommaschine. 1917. M. 10.35
- GRUBER (H.). Elektrotechnische Fachrechnen. 2te Auflage. 1919. M. 2.80
- GUIDI (C.). Lezioni sulla scienza delle costruzioni date nel r. Politecnico di Torino. Parte I: Nozioni di statica grafica. 9a edizione. Torino, tip. V. Bona, 1919. 8vo. 7 + 150 pp. L. 6.00
- IMMLER (W.). Flugzeugkompasswesen und Flugsteuerkunde. (Volkmann's Bibliothek für Flugwesen, Nr. 13.) 1918. M. 3.80
- KÖHN (P.). Elektrische Kraftübertragung. 2te Auflage. Leipzig, Teubner, 1919. Geb. M. 1.90
- LANGE (M.). Das Schachspiel und seine strategischen Prinzipien. 3te Auflage. Leipzig, Teubner, 1918. Geb. M. 1.90
- MARCH (A.). Theorie der Strahlung und der Quanten. Leipzig, Barth, 1919. M. 12.00

- MISES (R. VON). Vorträge über Theorie und Berechnung der Flugzeuge in elementarer Darstellung. Berlin, Springer, 1918. M. 8.00
- MÜLLER (E.). Lehrbuch der darstellenden Geometrie für technische Hochschulen. 2ter Band. 2te Auflage. Leipzig, Teubner, 1919. Geb. M. 5.40
- MURANI (O.). Lezioni di termodinamica, dettate nel Politecnico di Milano. (Biblioteca Tecnica.) Milano, Hoepli, 1919. 8vo. 11 + 220 pp. L. 9.50
- NUDORF (G.). Mathematische Chronologie. Stuttgart, Holland und Josenhans, 1914. 8 + 153 pp.
- PIETZSCH (H.). Festigkeitslehre und Materialkunde für das Flugwesen. (Volckmanns Bibliothek für Flugwesen, Nr. 7.) 1917. M. 3.70
- PUGLIONISI (S.). Praticità e sveltezza nei calcoli nautici: metodo pratico e tavole nautiche per calcolare la retta altezza e l'ora vera di bordo in pochi minuti . . . Riposto, tip. Dante Alighieri, 1918. 8vo. 83 pp. L. 7.50
- SCHWENGLER (J.). Die Statik im Flugzeugbau. (Bibliothek für Luftschiffahrt und Flugtechnik, Nr. 16.) 1917. M. 8.40
- SKOPNIK (O. L.). Festigkeitslehre für den Flugzeugbau. (Flugtechnische Bibliothek, Nr. 8.) 1919. M. 4.35
- WIENER (O.). Die streckenweise Berechnung der Geschossflugbahn. Leipzig, Teubner, 1919.
- WIESENT (J.). Die Fortschritte der drahtlosen Telegraphie und der physikalischen Grundlagen. Stuttgart, Enke, 1919.
- WULFF-PARCHIM (L.). Fragmente zur Theorie und Praxis der Kristalle. Selbstverlag. 1918.

The American Mathematical Monthly

OFFICIAL ORGAN OF

The Mathematical Association of America

**Is the Only Journal of Collegiate Grade in
The Mathematical Field in this Country**

This means that its mathematical contributions can be read and understood by those who have not specialized in mathematics beyond the Calculus.

The Historical Papers, which are numerous and of high grade, are based upon original research.

The Questions and Discussions, which are timely and interesting, cover a wide variety of topics.

Surveys of the contents of recent books and periodicals constitute a valuable guide to current mathematical literature.

The Topics of Undergraduate Mathematical Clubs have excited wide interest both in this Country and in Great Britain.

The Notes and News cover a wide range of interest and information both in this country and in foreign countries.

The Problems and Solutions hold the attention and activity of a large number of persons who are lovers of mathematics for its own sake.

There are other journals suited to the Secondary field, and there are still others of technical scientific character in the University field: but the MONTHLY is the only journal of Collegiate grade in America suited to the needs of the non-specialist in mathematics.

THE MATHEMATICAL ASSOCIATION OF AMERICA now has over eleven hundred individual and institutional members. There are already nine sections formed, representing twelve different states. The Association has held so far two national meetings per year, one in September and one in December. The sections, for the most part, hold two meetings each year. All meetings, both national and sectional, are reported in the Official Journal, and many of the papers presented at these meetings are published in full.

The slogan of the Association is to include in its membership every teacher of collegiate mathematics in America and to make such membership worth while. Application blanks for membership may be obtained from the Secretary, W. D. Cairns, 27 King Street, Oberlin, Ohio.

Whole No. 285

CONTENTS

	PAGE
Integro-Differential Equations with Constant Limits of Integration. By Dr. I. A. BARNETT - - - - -	193
On a Pencil of Nodal Cubics. By Professor NATHAN ALTSHILLER-COURT - - - - -	203
Definition and Illustrations of New Arithmetical Group Invariants. By Professor E. T. BELL - - - - -	211
Matrices and Determinoids. By Professor J. B. SHAW - - - - -	224
Notes - - - - -	233
New Publications - - - - -	237

Changes of address of members, exchanges, and subscribers should be communicated at once to the Secretary of the American Mathematical Society, 501 West 116th Street, New York.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to The American Mathematical Society, 41 North Queen St., Lancaster, Pa., or 501 West 116th Street, New York.

The initiation fees and annual dues of members of the American Mathematical Society are payable to the Treasurer of the American Mathematical Society, Professor J. H. TANNER, Cornell University, Ithaca, N. Y.

Articles for insertion in the BULLETIN should be addressed to the
BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.

The following dates have been fixed for the meetings of the Society :

New York: February 28, 1920. April 24, 1920

THE NEW ERA PRINT, LANCASTER, PA.

MAR 29 1920

BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY

**A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE**

EDITED BY

F. N. GOLE	VIRGIL SNYDER	J. W. YOUNG
ALEXANDER ZIWET	D. E. SMITH	
T. LEVI-CIVITA	R. C. ARCHIBALD	

VOLUME XXVI., NUMBER 6

MARCH, 1920

**PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK**

1920

Entered at the Post Office at Lancaster, Pa., as second class matter
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER

Whole No. 286

Digitized by Google **\$5.00 a year**

A Mathematical Correlation

Teachers who have recognized the difficulties encountered by pupils beginning algebra will welcome

GENERAL MATHEMATICS

By **R. SCHORLING**, *Columbia University*, and **W. B. REEVE**,
University of Minnesota

The purpose of this book is to give first-year high-school pupils a knowledge of the interrelation of algebra, geometry, trigonometry, practical drawing, and statistics. The text contains a wealth of material in the way of desirable options.



GINN AND COMPANY

70 Fifth Avenue

NEW YORK

THE TWENTY-SIXTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-sixth annual meeting of the Society was held at Columbia University on Tuesday and Wednesday, December 30-31, extending through two sessions on each day. The attendance included the following ninety-six members:

Dr. J. W. Alexander, Mr. P. L. Alger, Professor R. C. Archibald, Professor C. S. Atchison, Professor Clara L. Bacon, Dr. Charlotte C. Barnum, Professor A. A. Bennett, Professor E. G. Bill, Professor C. L. Bouton, Professor Joseph Bowden, Professor E. W. Brown, Professor Daniel Buchanan, Professor R. W. Burgess, Professor W. D. Cairns, Miss M. F. Chadbourne, Dr. Teresa Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Professor Elizabeth B. Cowley, Professor C. H. Currier, Dr. Mary F. Curtis, Dr. Tobias Dantzig, Professor J. V. DePorte, Professor C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor T. M. Focke, Professor C. H. Forsyth, Mr. T. C. Fry, Professor A. S. Gale, Professor W. V. N. Garretson, Professor O. E. Glenn, Professor W. C. Graustein, Dr. T. H. Gronwall, Professor H. E. Hawkes, Professor Olive C. Hazlett, Professor L. A. Howland, Professor W. A. Hurwitz, Professor Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. E. A. T. Kircher, Professor P. A. Lambert, Dr. K. W. Lamson, Professor D. D. Leib, Professor Florence P. Lewis, Professor P. H. Linehan, Professor Joseph Lipka, Professor C. R. MacInnes, Professor H. H. Mitchell, Professor C. N. Moore, Professor Frank Morley, Dr. H. C. M. Morse, Professor G. W. Mullins, Mr. F. H. Murray, Professor F. W. Owens, Dr. Helen B. Owens, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor Arthur Ranum, Professor H. W. Reddick, Dr. C. N. Reynolds, Jr., Mr. L. H. Rice, Professor R. G. D. Richardson, Dr. J. F. Ritt, Professor J. E. Rowe, Dr. Caroline E. Seely, Professor L. P. Siceloff, Dr. W. G. Simon, Professor C. G. Simpson, Professor Mary E. Sinclair, Professor H. E. Slaught, Professor Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Professor Sarah E. Smith, Professor W. M. Smith, Mr. J. J. Tanzola, Dr. J. S. Taylor, Mr. H. M. Terrill, Professor H. D. Thompson, Mr.

H. S. Vandiver, Professor Oswald Veblen, Mr. J. L. Walsh, Mr. H. E. Webb, Dr. Eula A. Weeks, Professor Mary E. Wells, Professor H. S. White, Professor E. E. Whitford, Dr. Norbert Wiener, Professor F. B. Wiley, Professor Ruth G. Wood, Professor J. W. Young.

President Frank Morley occupied the chair, being relieved at the last session by Professor J. L. Coolidge. The Council announced the election of the following persons to membership in the Society: Dr. H. E. Bray, Rice Institute; Professor I. L. Miller, Carthage College; Dr. Helen B. Owens, Cornell University; Professor E. W. Pehrson, University of Utah. Ten applications for membership in the Society were received.

The following resolutions, introduced by Professor R. C. Archibald as chairman of the committee on bibliography, were adopted by the Council:

1. The Council regards the preparation and publication, in America, of a dictionary of mathematical terms as not only most desirable but also entirely feasible, provided that financial aid for the preparation of the manuscript can be secured.

2. Impressed with possibilities for the more extensive development of pure and applied mathematics in America, and with the importance of such development to the nation, the Council records its conviction that there are undertakings whose active consideration would be highly desirable if adequate financial assistance might be regarded as available. Among such undertakings are: 1. The preparation and publication by societies or individuals of surveys, introductory monographs, translations, memoirs, and treatises, in important fields, including the history of mathematics. 2. The organization of research fellowships. 3. The preparation and publication of an encyclopaedia of mathematics in English. 4. The preparation and publication of an annual critical survey, in English, of the mathematical literature of the world. 5. The preparation and publication of a biographical and bibliographical dictionary of mathematicians.

It was decided to proceed with the incorporation of the Society under the general law of the State of New York. A committee was appointed to consider plans for the organization and administration of the Society after the retirement of the present Secretary and Librarian from their offices at the close of the present year. A committee was also appointed to

consider the formation of an international union of mathematicians. The committee on mathematical requirements presented a report, which was laid over for consideration at the February meeting.

The total membership of the Society is now 733, including 80 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 393; the number of papers read was 187. The number of members attending at least one meeting during the year was 252. At the annual election 156 votes were cast. The Treasurer's report shows a balance of \$10,692.23, including the life-membership fund of \$7,168.87. Sales of the Society's publications during the year amounted to \$1,811.52. The Library now contains 5,690 volumes, excluding some 500 unbound dissertations.

At the annual election, which closed on Wednesday morning, the following officers and other members of the Council were chosen:

<i>Vice-Presidents,</i>	Professor C. N. HASKINS, Professor R. G. D. RICHARDSON.
<i>Secretary,</i>	Professor F. N. COLE.
<i>Treasurer,</i>	Professor J. H. TANNER.
<i>Librarian,</i>	Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to serve until December, 1922,

Professor T. H. HILDEBRANDT,	Professor W. A. MANNING,
Professor EDWARD KASNER,	Professor H. H. MITCHELL.

The meeting of the Society immediately preceded that of the Mathematical Association of America on January 1-2. A very pleasant occasion was the joint dinner of the two organizations on New Year's eve with an attendance of 114 members and friends.

The following papers were read at the annual meeting:

- (1) Dr. H. F. MACNEISH: "The sum of the face angles of a polyhedron in space of n dimensions."

(2) Dr. G. A. PFEIFFER: "A connected set of points which contains no continuous arc."

(3) Professor C. J. KEYSER: "Fundamental types of groups of relations of an infinite field."

(4) Professor JOSEPH LIPKA: "The theorem of Thomson and Tait and its converse in space of n dimensions."

(5) Professor A. A. BENNETT: "Poncelet polygons in higher space."

(6) Professor A. A. BENNETT: "Continuous matrices, algebraic correspondences, and closure."

(7) Professor L. A. HOWLAND: "Concerning points of inflection on a rational plane quartic."

(8) Dr. H. C. M. MORSE: "Geodesics motion on a surface of negative curvature."

(9) Professor J. L. COOLIDGE: "The geometry of hermitian forms."

(10) Professors H. B. PHILLIPS and C. L. E. MOORE: "Rotations in space of even dimensions."

(11) Professors C. L. E. MOORE and H. B. PHILLIPS: "Note on geometric products."

(12) Professor O. E. GLENN: "A memoir upon formal invariancy with regard to binary modular transformations."

(13) Professor O. E. GLENN: "The invariant problem of the relativity transformations of Lorentz appertaining to the mutual attraction of two material points" (preliminary report).

(14) Dr. NORBERT WIENER: "The mean of a functional of arbitrary elements."

(15) Dr. NORBERT WIENER: "Bilinear operations generating all operations rational in a domain Ω ."

(16) Dr. NORBERT WIENER: "Fréchet's calcul fonctionnel and analysis situs."

(17) Dr. NORBERT WIENER: "A set of postulates for fields."

(18) Mr. J. L. WALSH: "On the location of the roots of the jacobian of two binary forms, and of the derivative of a rational function."

(19) Mr. J. L. WALSH: "On the proof of Cauchy's integral formula by means of Green's formula."

(20) Professor DUNHAM JACKSON: "On the order of magnitude of the coefficients in trigonometric interpolation."

(21) Mr. P. L. ALGER: "A problem of electrical engineering."

(22) Professor W. B. FITE: "Properties of the solutions of certain functional differential equations."

(23) Professor W. C. GRAUSTEIN: "Determination of the pairs of ordered real points representing a complex point."

(24) Dr. J. S. TAYLOR: "Sheffer's set of five postulates for Boolean algebras in terms of the operation 'rejection' made completely independent."

(25) Professor C. H. FORSYTH: " ${}_nE_x$, the magic wand of actuarial theory."

(26) Professor C. H. FORSYTH: "A formula for determining the mode of a frequency distribution."

(27) Professor DANIEL BUCHANAN: "Asymptotic orbits near the equilateral triangle equilibrium points in the problem of three finite bodies."

(28) Professor C. L. BOUTON: "The definition of birational transformations by means of differential equations."

(29) Professor W. C. GRAUSTEIN: "Area-preserving, parallel maps in relation to translation surfaces."

(30) Dr. C. N. REYNOLDS, Jr.: "Note on linear differential equations of the fourth order whose solutions satisfy a homogeneous quadratic identity."

(31) Professor J. E. ROWE: "A practical problem of aerodynamics and thermodynamics."

(32) Professor W. B. CARVER and Mrs. E. F. KING: "A property of permutation groups analogous to multiple transitivity."

(33) Professor OLIVE C. HAZLETT: "Some pseudo-finiteness theorems in the general theory of modular covariants."

(34) Dr. MARY F. CURTIS: "Note on the rectifiability of a twisted cubic."

(35) Dr. TERESA COHEN: "The representation of fractions of periods on algebraic curves by means of virtual point sets."

(36) Professor C. A. FISCHER: "Necessary and sufficient conditions that a linear transformation be completely continuous."

(37) Dr. S. D. ZELDIN: "On the structure of finite continuous groups with a single exceptional infinitesimal transformation."

(38) Mr. J. L. WALSH: "On the location of the roots of the derivative of a polynomial."

Dr. Zeldin was introduced by Professor Lipka. The paper of Dr. MacNeish, the first paper of Professor Bennett, the

papers of Professors Phillips and Moore, the third and fourth papers of Dr. Wiener, the first paper of Professor Forsyth, and the paper of Professor Carver and Mrs. King were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dr. MacNeish obtains the following formula:

$$\Sigma = \left[\frac{2}{3}n(n-1)a_0 - \frac{8}{n-2}a_3 \right]$$

right angles for the sum of the face angles of the plane faces of a simple polyhedron in n -space, where a_r ($r = 0, 1, \dots, n-1$) represents the number of r -space faces. A simple polyhedron in n -space is defined as a polyhedron in which $n-iC_{r-i}$ of its r -space elements meet at each i -space ($r > i$; $i = 0, 1, \dots, n-2$; $r = 1, 2, \dots, n-1$). If more than $n-iC_{r-i}$ of the r -space elements meet at any i -space, the polyhedron is called a multiple polyhedron. Formulas are developed for the sum of the face angles of certain types of multiple polyhedrons.

2. In this paper Dr. Pfeiffer presents an example of a connected set of points which contains no continuous arc. This example is the set of points (x, y) such that

$$y = \sum_1^{\infty} c_n \phi(x - \omega_n) \quad \left(-\frac{1}{2} < x < \frac{1}{2}\right),$$

where

$$\phi(x - \omega_n) = \sin \frac{1}{x - \omega_n},$$

$x \neq \omega_n$ and $-1 - \omega_n < x < 1 - \omega_n$, $\phi(0) = 0$; ω_n is the abscissa of the n th rational point of the interval $-\frac{1}{2} < x < \frac{1}{2}$ with respect to a definite enumeration of the rational points of that interval and $c_1, c_2, c_3, \dots$ is a denumerable set of positive numbers such that $\sum_1^{\infty} c_n$ is convergent.

3. In Professor Keyser's paper the relations of an infinite field (coincident with the domain and the codomain) are

viewed as falling under nine types: $1 - 1$, $1 - s$, $s - 1$, $s - s$, $1 - m$, $m - 1$, $s - m$, $m - s$, $m - m$, where m means many (i. e., two or more) and s means some in the sense that a relation whose type symbol involves s contains at least one couple in which s is represented by only one term and at least one couple in which s is represented by more than one term. A class of relations containing one or more relations of each of k of the nine types, as $t_1, t_2, \dots, t_k$, is said to be of the type $t_1 - t_2 - \dots - t_k$. There thus arise 511 class types, called fundamental types. Among the infinitude of classes of a given type there is one class which includes every subclass of the same type. This distinguished class is said to be a fundamental class. Accordingly there are 511 fundamental classes, one for each fundamental type. Given a rule by which any two relations of the field can be combined so as to yield a relation of the field, there arise two problems: (1) To determine which of the fundamental classes are groups under the rule; (2) To determine which of the fundamental classes contain group subclasses of the same type as that of the containing class. The solution of (1), the rule being that of relative multiplication, was given by the author in a paper presented at the last April meeting of the Society. The present paper gives the solution of (2) for the same rule, and the solutions of both problems where the rule of combination is that of logical addition of relations.

4. The theorem of Thomson and Tait for space of n dimensions may be stated as follows: if from all points of an arbitrary hypersurface (space of $n - 1$ dimensions) particles not mutually influencing one another be projected normally in a conservative field of force, points which they reach with equal actions lie on a hypersurface cutting the paths at right angles. The paths form a system of $\infty^{2(n-1)}$ trajectories for each value of the constant of energy. The converse theorem proved by Professor Lipka states: if a system of $\infty^{2(n-1)}$ trajectories in space of n dimensions is such that any ∞^{n-1} trajectories of the system which meet an arbitrary hypersurface (space of $n - 1$ dimensions) orthogonally admit of ∞^1 orthogonal hypersurfaces, then the system may be considered as the trajectories in a conservative field of force. The purely geometric part of the theorem is true for all *natural* families of

curves. The n -dimensional spaces considered have constant curvature.*

5. Professor Bennett's first paper appears in full in the present number of the BULLETIN.

6. This paper by Professor Bennett shows how an algebraic correspondence may be viewed as a matrix with a continuous array of elements. The result of two successive correspondences is represented by the matrix obtained with the usual rules for a product. The sum of two matrices has an analogous interpretation in correspondences. The matrix representation emphasizes a weighting of correspondences which is suggested, but less readily, by geometric considerations. The meaning of the inverse of reducible correspondences becomes precise only when weights are assigned in the manner required by the matrix representations. The inverse of a weighted algebraic correspondence may fail to exist altogether or may not be algebraic when these terms are carefully defined. The question of the algebraic character of an inverse that exists is entirely equivalent to the closure of the original correspondence, and to the existence of variable closed algebraic configurations on the fundamental manifold.

7. It is well known that the six inflexions of a rational plane quartic lie on a conic. To the best knowledge of the author, however, there is no proof of this theorem which is at once complete and also based on elementary considerations. Professor Howland's paper attempts such a proof, and follows that by special consideration of the cases of a quartic with a triple point.

8. The study of dynamical systems leads at once to the study of geodesics as a type of motion. A primary purpose of Dr. Morse's study of geodesics was to establish the existence of a class of recurrent motions called discontinuous recurrent motions by Professor Birkhoff, and to examine a case where, if the recurrent motions, not simply periodic, existed at all, they would furnish a very general example of such motions. The surfaces of negative curvature considered are surfaces

* For a discussion of these theorems for space of three dimensions see Edward Kasner, *Trans. Amer. Math. Society*, vol. 11 (1910), pp. 121-140.

which may be cut up into n simply connected portions (n arbitrary, and exceeding one) none of which lie wholly in any finite portion of space, together with one portion S which lies wholly in a finite portion of space, and which may be of arbitrary connectivity exceeding two. Geodesics are considered which if continued indefinitely in either sense lie wholly on a finite portion of the surface. A fundamental result of this paper is a one-to-one representation of these geodesics in terms of the $2p$ fundamental contours of S , together with the n closed curves bounding S and separating S from the remaining portions of the original surface. A second paper will apply the results of this paper to the theory of dynamical systems.

9. In Professor Coolidge's paper the problem of reducing a hermitian form of non-vanishing discriminant is discussed. Special attention is paid to those collineations of n -space which leave invariant the form $\sum x_i \bar{x}_i$ ($i = 0, 1, 2, \dots, n$).

10. Professors Phillips and Moore treat rotations in space of $2n$ dimensions by means of the Gibbs dyadic, deriving in particular a form

$$\phi = e^{I \cdot M} = e^{I \cdot \sum q_i M_i}$$

for such a rotation. Here M is a complex two-vector, M_i a set of completely perpendicular plane vectors, and q_i the angles of rotation in those planes, I being the identical dyadic and $I \cdot M$ the dot product of Lewis. A sufficient condition that two rotations ϕ and ϕ' be commutative is that the vector of the dyadic $(I \cdot M) (I \cdot M')$ be zero. This is also necessary unless two or more of the q 's equal $\pm \pi$.

In four dimensions any rotation can be expressed as the product of two equiangular rotations

$$e^{qI \cdot (M_1 + M_2)} \quad \text{and} \quad e^{q'I \cdot (M_1 - M_2)}.$$

These equiangular rotations have the property of rotating all vectors through the same angle and are analogous to the rotations resulting from the multiplication of quaternions referred in the first case to right-hand axes, in the second to left-hand ones.

11. Products which in terms of units have identities of the form $k_{12}k_{34} = k_{1234}$ (outer product of Grassmann) and $k_1k_{12} = k_2$

(inner product) are well known. Professors Moore and Phillips consider such products as $k_{12}k_{13} = k_{23}$, this particular one being an extension of Gibbs' cross product. These occur in the analytic expression of complete perpendicularity and in the study of commutative rotations.

12. The memoir of Professor Glenn comprises a general exposition relating to the following subdivisions: Invariants belonging to prescribed domains of transformations of finite order on n variables. A complete system under the above theory, in the $GF(p^2)$, of a binary quantic of order m . Systems of universal covariants of the groups $G_x: x_1 \equiv x_1' + x_2', x_2 \equiv x_2' \pmod{p}$, $H_x: x_1 \equiv x_1' + x_2' + x_3', x_3 \equiv x_3' \pmod{p}$, and of the simultaneous groups G_x, G_y . A theorem upon modular systems according to which the formal concomitants modulo p of a binary m -ic are constructed as simultaneous systems of sets of covariants of lower order. Development of a fundamental system of covariants for a linear form and a quadratic modulo 2, of a single binary quartic modulo 2, and of a complete system of semivariants of the quartic modulo 3. Somewhat extensive data concerning a complete system of covariants of the quartic modulo 3 are also included. The last section of the paper gives a theory of complete systems under the transformations on the velocities and accelerations which figure in the theory of relativity of motion.

13. The components F_x, F_y, F_z in the theory of the mutual attractions of two material points are invariantive under the linear transformations S on four variables, called by Lorentz the general transformations of relativity (time being a fourth dimension). A theory of concomitants in the domain of the coefficients of S , based upon the equations of the four planes upon its poles, is treated in Professor Glenn's paper.

14. Dr. Wiener shows how, by applying to E. H. Moore's notion of development the notion of a weighting (i.e., of a measure of volume), a type of mean can be obtained which, under certain very general hypotheses, will be an example of the generalized integration of P. J. Daniell. The means of a function of n variables and of a function of a discrete infinity of variables are shown to be examples of this concept, while it is proved that by a proper "weighting" the mean or

integral of a function of a line can be defined. In this connection, the concept of the "coefficient of irregularity" of a function is introduced, viz.,

$$\text{Max} \frac{\Delta f(x)}{\Delta(x)},$$

and it is proved that a bounded set of functions whose coefficients of irregularity are all less than some finite quantity is extremal in the sense of Fréchet.

15. An operation $f(x_1, x_2, \dots, x_p)$ is said to generate a class K of operations when

(1) K contains f ;

(2) if K contains $g(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_k)$, it also contains $g(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{j-1}, x_i, x_{j+1}, \dots, x_k)$ and $g(x_1, x_2, \dots, x_{i-1}, x_j, x_{i+1}, \dots, x_{j-1}, x_i, x_{j+1}, \dots, x_k)$.

(3) if K contains $g(x_1, x_2, \dots, x_k)$ and $h(x'_1, x'_2, \dots, x'_l)$, K also contains the operation on $x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_k, x'_1, x'_2, \dots, x'_l$ formed by substituting h for x_i in g .

(4) K contains no proper subset with the three preceding properties.

Dr. Wiener proves that the operations that are bilinear and generate the set of all rational operations with rational coefficients may be reduced by a rational linear transformation to one of the forms

$$\frac{x-y}{x+Gy} \quad (G \text{ rational and } \neq -1)$$

$$\frac{n(x-y+xy)}{ny+xy} \quad (n \text{ integral and } \neq 0),$$

and that every such operation will generate all rational operations with rational coefficients. The results are extended in part to other domains of rationality.

16. Dr. Wiener defines continuous deformations and their limits on the basis of the limit of a sequence of points. The notion of a one-parameter family of deformations follows readily, and an n -space can be defined by employing this concept together with a process of mathematical induction.

Certain properties of the n -spaces selected from a given space are discussed, and finally a categorical set of postulates is given for the analysis situs of a space of any finite number of dimensions.

17. Dr. Wiener shows that all algebraic operations are obtainable by the iteration of $x|y = 1 - x/y$. A set of postulates is derived for this operation. This set is consistent, independent, and equivalent to the sets given by E. V. Huntington for fields. It is also shown that every operation and concept in complex algebra can be derived from the operation $1 - K(x/y)$, where K stands for "conjugate of."

18. Mr. Walsh's paper is a continuation of a recent paper published in the *Transactions* under the same title. It is proved as a lemma that if three variable points of the complex plane determine a fourth variable point by a real constant cross ratio with those three, and if the three points have as their respective envelopes closed regions bounded by circles, then the envelope of the fourth point is a fourth closed region bounded by a circle. This lemma is applied in proving that if three circular regions contain respectively k roots of a binary form of degree p_1 , the remaining $p_1 - k$ roots of this form, and all the roots of a second form of degree p_2 , then these three circular regions and the fourth circular region of the lemma corresponding to the cross ratio p_1/k contain all the roots of the jacobian of the two forms. If the four regions have no common point, they contain respectively the following numbers of roots of the jacobian: $k - 1$, $p_1 - k - 1$, $p_2 - 1$, and 1.

19. Mr. Walsh's second paper appeared in full in the January BULLETIN.

20. It is well known that there is a remarkable and far-reaching parallelism between the properties of the trigonometric interpolating formula determined by the values of a given function at a set of equally spaced points, and those of the formula of approximation obtained by taking the first terms of the Fourier's series for the same function. This parallelism is closer than would be indicated at first sight by the correspondence of sums and integrals in the formulas for the coefficient of the two approximating functions. It has

to be regarded as due in part to the special properties of the trigonometric functions, not merely to the definition of an integral as the limit of a sum. Professor Jackson's paper is concerned with the order of magnitude of the coefficients in the interpolating formula, in case the given function is of limited variation, or has a derivative of limited variation. The method of partial summation is used to give results analogous to those which Picard (*Traité d'Analyse*, volume 1) obtains for the Fourier's series by the second law of the mean and integration by parts. Incidentally, in the case of a derivative of limited variation, a relatively simple proof is obtained for the convergence of the Fourier's series, as well as of the interpolating formula, to the value of the given function.

21. Mr. Alger presents an unsolved problem of electrical engineering which is of considerable practical interest to engineers. The problem is to determine the distribution of magnetic flux and current density across the thickness of an infinite plane lamina of iron, when an harmonically varying magnetizing force is applied. Under the assumption of uniform permeability, the problem has been completely solved, and its solution may be found in any standard advanced text book of electrical engineering. The solution is in this case analogous to that of the flow of heat in a plane, and the flux and current densities at any point are proportional to the hyperbolic functions of a complex quantity. In actual practice, however, the permeability varies over a wide range and is directly dependent on the magnetizing force. The solution of the problem, when the permeability is assumed variable, has never been obtained, and the whole problem under this assumption is so little understood that the simplest qualitative results of a mathematical study of it should prove valuable. When the permeability is variable, the harmonic impressed magnetizing force sets up magnetic fluxes of all odd multiples of the impressed frequency, and the solution of the problem involves the determination of the coefficients in a Fourier's series. The law of variation of permeability is a purely empirical one, so that one difficulty in the problem is the choice of a law of variation that will make a solution possible while it still sufficiently represents the facts.

22. In this paper Professor Fite discusses the oscillatory properties of the solutions of binary functional differential equations of the form $y^{(n)}(x) + r(x)y(k-x) = 0$, where $|r(x)| > b > 0$ for all finite values of x . He shows that when n is odd every solution changes sign an infinite number of times, but that when n is even a solution may change sign only a finite number of times. Whether this finite number is odd or even depends upon the sign of $r(x)$. He also determines certain necessary conditions in order that these equations, for $n \leq 2$, may have odd or even solutions.

23. A complex point P in a complex space of n dimensions, since it depends on $2n$ real parameters, may be represented by a pair of real points P_1, P_2 in the space. It is reasonable to restrict the choice of P_1, P_2 so that (a) they coincide in P , if P is a real point; (b) they are the same for the point $\bar{P}$, conjugate to P , as they are for P , and in one order represent P and in the opposite order $\bar{P}$; (c) their coordinates, referred to a cartesian system, are analytic functions of the coordinates of P and $\bar{P}$, referred to this system; (d) their transforms P_1', P_2' by an arbitrary transformation of a chosen group of real point transformations represent the point P' into which P is carried by the transformation. Professor Graustein determines, for the group of real collineations and for each of the more important of its subgroups, the affine group, the group of similarity transformations, the group of motions and the group of translations, all the ordered point pairs P_1, P_2 , which represent the given point P and have the desired properties. The determination, in the case of each group, depends on the solution of a functional equation.

24. As has been shown by L. L. Dines, Sheffer's five postulates for Boolean algebras in terms of the operation "rejection," while independent in the sense that no one of them is implied by the other four, are not completely independent in the sense defined by Professor E. H. Moore, in as much as the negative of the first postulate implies the third, fourth, and fifth. Dr. Taylor's paper demonstrates the fact that the postulates become completely independent if the first postulate is replaced by one postulating a minimum of four distinct elements instead of two.

25. The present value of a pure endowment, or ${}_nE_x$, seems to be generally regarded as on a par in importance with quite a number of other actuarial symbols of similar nature. Professor Forsyth shows how the symbol may be used to affect ready analyses of numerous situations in actuarial theory which otherwise prove subtly complicated. In particular, he gives concise derivations of formulas used in present day methods of valuation of life insurance policies—such as full preliminary term, modified preliminary term, etc.—wherein the conciseness is due directly to the use of the symbol ${}_nE_x$.

26. The main disadvantage in the use of the mode as an average is the difficulty met in its determination when the statistical data are given in the form of a frequency distribution when the frequencies correspond to class-marks. Professor Forsyth derives a formula for determining the mode in situations like those just mentioned, based upon his formula used for interpolation of ordinates among areas presented in the December (1916) issue of the *Quarterly Publications of the American Statistical Association*.

27. In a previous paper, presented to the Society in December, 1918, Professor Buchanan discussed the orbits which are asymptotic to the straight line equilibrium points in the problem of three finite bodies. The present paper deals with the corresponding orbits for the equilateral triangle equilibrium points.

28. This note shows how to set up certain systems of partial differential equations whose solutions establish all the birational transformations in space of n dimensions. Let $x_1, x_2, \dots, x_{n+1}$ be homogeneous coordinates. The transformation $\rho x_i' = \varphi_i(x_1, x_2, \dots, x_{n+1})$ ($i = 1, 2, \dots, n+1$) is birational if the functions φ_i are homogeneous polynomials of the same degree, m , and if the inverse transformation has the same form, its polynomials being of degree m' . Professor Bouton begins by reducing the transformation by means of projections to a normal form in which $\varphi_1 = x_1^m + x_1^{m-2}A_{12} + \dots + A_{1m}$, $\varphi_i = x_i x_1^{m-1} + x_1^{m-2}A_{i2} + \dots + A_{im}$ ($i = 2, 3, \dots, n+1$), where A_{ik} is a homogeneous polynomial in $x_2, \dots, x_{n+1}$, of degree k , and with the inverse in similar form. The differential equations established are those satisfied by

the polynomials A_{i1} . For example, if $m = m' = 2$, the A_{i2} are solutions of the system of partial differential equations

$$2x_i A_{i2} + \sum_{s=2}^{n+1} A_{s2} \frac{\partial A_{i2}}{\partial x_s} = 0 \quad (i = 2, 3, \dots, n+1).$$

Conversely, any set of solutions of this system, which are homogeneous polynomials of degree two, will yield a birational transformation, which may, however, be reducible to a projection. There seem to be no references in the literature to this method of setting up all the birational transformations in space of n dimensions.

29. Two real surfaces, in continuous one-to-one point correspondence, with the directed normals in corresponding points parallel, may be said to correspond by a parallel map, or more specifically, by a directly parallel or an inversely parallel map, according as corresponding directions of rotation about corresponding points are the same or opposite. Professor Graustein shows that two real surfaces, which correspond by an inversely parallel, area-preserving map of a particular kind, are translation surfaces with real generators, whereas, if the map is directly parallel and area-preserving of a certain type, the surfaces are translation surfaces with conjugate-imaginary generators. In particular, two real surfaces, applicable by an inversely parallel map, are translation surfaces with real perpendicular generators and hence cylinders; if, on the other hand, the surfaces are applicable by a directly parallel map with the property that the angle between corresponding curves in corresponding points is constant, not 0 or π , they are translation surfaces with conjugate-imaginary minimal curves as generators and are, then, minimal surfaces.

30. Dr. Reynolds proves that if a given linear differential equation of the fourth order has four linearly independent real solutions which satisfy a non-singular quadratic identity, then it may be reduced to a form which may be said to be self-adjoint with respect to the third row of the wronskian of any four linearly independent solutions. The signature of the quadratic identity associated with the equation is then evaluated in terms of the coefficients of the given equation and several theorems concerning the zeros of such an equation are proved.

31. The purpose of Professor Rowe's paper was to make a mathematical statement of the aeronautical radiator problem.

32. Professor Carver and Mrs. King define a property of permutation groups, closely analogous to multiple transitivity, as follows:

A permutation group of degree n is said to be k -ply pseudo-transitive, $1 \leq k \leq n - 1$, if there is a set of k letters such that there is at least one permutation of the group sending this set into any arbitrarily chosen set of k letters in *some order*. It is evident that the case $k = 1$ is the same as ordinary simple transitivity; that if a group is k -ply transitive it is k -ply pseudo-transitive, but not conversely; and that if a group of degree n is k -ply pseudo-transitive it is also $(n - k)$ -ply pseudo-transitive. Several theorems analogous to those for ordinary multiple transitivity are readily obtained.

Among the groups of degree equal to or less than nine, there are just seven cases of groups having multiple pseudo-transitivity of a multiplicity other than is obviously implied by their ordinary multiple transitivity.

The notion has an incidental application to, and was suggested by, a question concerning the condition for apolarity between two binary forms.

33. In any theory of covariants, we find that closely related to the point of view taken when we try to express all covariants as polynomials in a finite number of covariants is the one taken when we endeavor to express all covariants as rational (though not necessarily integral) functions of a finite number of covariants. Professor Hazlett's paper attacks modular covariants of a single binary form from this second point of view by employing a method which has its origin in Hermite's fundamental memoir on associated forms. The main theorem of the present paper proves that, if f is a binary form of order not congruent to zero modulo p , then any modular covariant of f for the $GF[p^n]$ is expressible (aside from a power of f) as a polynomial in Q and L , where the coefficients of the terms in Q are polynomials in the forms associated with f . From this theorem flow several corollaries, of which one gives a neat method of constructing a modular covariant having a given leader. This theorem, together with the corollaries, is verified for the binary quadratic, modulo 3.

34. Dr. Curtis's paper appears in full in the present number of the BULLETIN.

35. Suppose a curve C of order m and genus p with d double points. Miss Cohen considers the linear series of curves of order αn cutting C n times at each of $\alpha m - p$ fixed points, among which are the d double points, the value of α for m odd being $\frac{1}{2}(m - 1)$, and for m even, $\frac{1}{2}(m - 2)$. There are exactly n^{2p} such curves cutting C n times at p further points. Suppose such a set of p points to be $A_1, A_2, \dots, A_p$, the fixed base points to be $F_1, F_2, \dots, F_{\alpha m - p}$, and any line section represented by L . Then $A_1 + \dots + A_p + F_1 + \dots + F_{\alpha m - p} - \alpha L$ is a virtual set that has the properties of an n th of a period for a given set of $\alpha m - p$ base points and the totality of such sets can be shown to represent the totality of n ths of periods. For p sufficiently high it is possible to put a curve of order α on $F_1, \dots, F_{\alpha m - p}$ with p residual intersections $P_1, P_2, \dots, P_p$. The n th of a period can then be represented by $A_1 + \dots + A_p - (P_1 + \dots + P_p)$.

The simplest case is presented by the elliptic parameters on the general cubic.

36. F. Riesz has derived a large part of the Fredholm theory of integral equations for the equation

$$h(x) = f(x) + \lambda T(f),$$

where $T(f)$ is a completely continuous, linear transformation, and he has also proved that every linear transformation, that is to say every linear functional depending on a parameter, can be expressed in terms of a Stieltjes integral such as

$$T(f) = \int_a^b f(y) d_y K(x, y).$$

In the present paper Professor Fischer has proved that if such a transformation is to be completely continuous, it is necessary and sufficient that the variation of $K(x, y)$ in y shall be a bounded function of x , and that whenever $x_1, x_2, \dots$ are chosen in such a way that $K(x_n, y)$ approaches a unique function of y when n becomes infinite, the variation of $[K(x_n, y) - \text{limit } K(x_n, y)]$ in y shall approach zero. Another necessary and sufficient condition is also found and proved equivalent to this one.

37. The problem of the structure of finite continuous groups consists in finding the structural constants determined by the alternants of operators, generating a group with r essential parameters, of the form

$$X_i = \sum_{k=1}^n \xi_k(x) \frac{\partial}{\partial x_k} \quad (i = 1, 2, \dots, r).$$

These alternants have the form

$$(X_i, X_j) = \sum_{s=1}^r c_{ijs} X_s \quad (i, j = 1, 2, \dots, r).$$

The c 's which are the structural constants must satisfy two conditions:

$$(1) \quad c_{ijk} = -c_{jik},$$

$$(2) \quad \sum_{i=1}^r (c_{ijk} c_{kls} + c_{jlk} c_{kls} + c_{lik} c_{kjs}) = 0 \quad (k = 1, 2, \dots, r).$$

Dr. Zeldin discusses the conditions to be imposed on the group with one exceptional infinitesimal transformation X_r , which would make all c_{irk} equal to zero, where $(i, k = 1, 2, \dots, r)$.

38. Jensen has recently stated the theorem that if $f(z)$ is a real polynomial and if there are described circles having as diameters the segments joining the pairs of conjugate imaginary roots of $f(z)$, then no non-real root of $f'(z)$ lies outside those circles. Mr. Walsh shows that on any segment of the axis of reals exterior to all of those circles and containing no root of $f(z)$ there is at most one root of $f'(z)$. If any of those circles has on or within it k roots of $f(z)$ and is not interior to nor has a point in common with any other of those circles, it has on or within it not more than $k + 1$ nor less than $k - 1$ roots of $f'(z)$. The paper contains also various other theorems connected with Jensen's theorem.

F. N. COLE,
Secretary.

THE ST. LOUIS MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE thirteenth western meeting of the American Mathematical Society, combining the forty-fourth regular meeting of the Chicago Section, and the twelfth regular meeting of the Southwestern Section, was held at the Soldan High School, St. Louis, Missouri, on Tuesday and Wednesday, December 30 and 31. On Tuesday afternoon there was a joint session with Section A of the American Association for the Advancement of Science, and the Missouri Section of the Mathematical Association of America.

The attendance at this meeting included fifty persons, among whom were the following forty-four members of the society:

Professor N. Altshiller-Court, Professor C. H. Ashton, Professor G. D. Birkhoff, Professor G. A. Bliss, Professor H. Blumberg, Professor W. C. Brenke, Dr. C. C. Camp, Professor A. B. Coble, Professor S. C. Davisson, Professor E. L. Dodd, Professor A. Dresden, Professor O. Dunkel, Professor H. J. Ettlinger, Professor Tomlinson Fort, Dr. J. B. Glasgow, Professor H. Halperin, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor G. O. James, Professor H. E. Jordan, Professor O. D. Kellogg, Professor Solomon Lefschetz, Professor A. C. Lunn, Dr. G. F. McEwen, Professor J. V. McKelvey, Professor T. E. Mason, Professor G. A. Miller, Professor B. E. Mitchell, Professor U. G. Mitchell, Professor F. R. Moulton, Dr. J. R. Musselman, Dr. C. A. Nelson, Professor B. L. Newkirk, Professor P. R. Rider, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Mr. J. B. Rosenbach, Dr. G. W. Smith, Professor R. B. Stone, Professor E. B. Stouffer, Dr. E. H. Taylor, Professor W. D. A. Westfall, Professor K. P. Williams.

At the dinner at the American Annex Hotel on Tuesday evening forty-eight persons were present.

At the business meeting of the Chicago Section and the Southwestern Section held at the end of the morning session of Wednesday, the following action was taken:

It was decided to hold the next meeting of the Southwestern Section at the University of Nebraska on November 27. The

following programme committee was elected: W. C. Brenke, chairman, H. L. Rietz, O. D. Kellogg, secretary.

The Chicago Section elected for the ensuing biennium, Professor R. D. Carmichael, chairman; Professor Arnold Dresden, secretary; Professor C. N. Moore, third member of programme committee.

A resolution thanking Professor E. J. Moulton for his services as acting secretary of the Chicago Section during 1918-19 was unanimously passed.

A resolution on the death of Professor Laenas Gifford Weld on November 28, 1919, was adopted by a rising vote.

The Chicago Section and the Southwestern Section passed a resolution expressing interest in an investigation of the possibility of cooperative work in pure mathematics and in the domain between mathematics and the related sciences, as suggested by the National Research Council.

At the joint meeting on Tuesday afternoon, at which Professor O. D. Kellogg, vice-president of Section A, presided, the following papers were read:

I. Professor G. D. BIRKHOFF, retiring vice-president of Section A of the American Association for the Advancement of Science, "Recent progress in dynamics."

II. Professor G. A. BLISS, retiring chairman of the Chicago Section of the American Mathematical Society, "Some recent developments in the calculus of variations."

III. Mr. H. H. PLATT, "A suggestion for the utilization of atmospheric molecular energy."

The sessions of Tuesday forenoon, Wednesday forenoon and afternoon were presided over by Professors Bliss, Birkhoff and Roeber respectively. At these sessions the following papers were read:

(1) Professor C. E. LOVE: "Note on a class of singular integral equations."

(2) Dr. R. F. BORDEN: "On the adjoint of a certain mixed equation."

(3) Professor G. O. JAMES: "On a philosophical aspect of the relativity theory of gravitation."

(4) Professor O. SCHMIEDEL: "On the early history of the number concept."

(5) Mr. B. Z. LINFIELD: "On the relation of the roots of a polynomial to the roots of its derivative."

(6) Professor E. L. DODD: "A comparison of the arithmetic mean with other weighted means under certain laws of error."

(7) Professor S. LEFSCHETZ: "Real abelian varieties."

(8) Professor N. ALTSHILLER-COURT: "On a pencil of nodal cubics."

(9) Professor N. ALTSHILLER-COURT: "On an orthocentric quadrilateral."

(10) Professor G. A. MILLER: "Properties of the subgroups of an abelian prime power group which are conjugate under the group of isomorphisms."

(11) Professor G. A. MILLER: "Groups generated by two operators of order three whose product is of order four."

(12) Professor H. L. RIETZ: "Urn schemata as a basis for the development of correlation theory."

(13) Professor K. P. WILLIAMS: "Concerning a problem in the flow of heat."

(14) Professor TOMLINSON FORT: "Note on Dirichlet and factorial series."

(15) Professor H. J. ETTLINGER: "Boundary value problems with regular singular points."

(16) Professor H. J. ETTLINGER: "Extension of an existence theorem for a non-self-adjoint boundary value problem."

(17) Professor G. C. EVANS: "Divergent Fourier series and integral equations."

(18) Mr. L. L. STEIMLEY: "On the general class of series of the form $Y(x) = c_0 + \sum c_n g(nx)$."

(19) Professor A. C. LUNN: "A principle of duality in thermodynamics."

(20) Professor R. L. BORGER: "On the Cauchy-Goursat theorem."

(21) Professor A. B. COBLE: "Some irrational and transcendental aspects of the invariants of a ternary quartic."

(22) Professor A. B. COBLE: "The Cayley symmetroid and the abelian modular functions of genus four."

(23) Professor E. J. WILCZYNSKI: "One-parameter families and nets of ruled surfaces and a new theory of congruences."

(24) Professor A. EMCH: "Intersections of straight lines and systems of plane algebraic curves and the surfaces generated by them."

(25) Professor PIERRE BOUTROUX: "A family of multiform functions."

(26) Dr. G. F. McEWEN: "Fluid temperature distributions

deduced from sources, boundary conditions, fluid velocity and convection."

(27) Dr. G. F. McEWEN: "Direct application of the idea of "random sampling" in estimating the precision of averages, and the significance of differences."

Dr. Borden was introduced by Professor Carmichael. The papers of Professor Love, Dr. Borden, Professor Schmiedel, Mr. Linfield, Professor Evans, Mr. Steimley, Professor Borger, Professor Wilczynski, Professor Emch, and Professor Boutroux were read by title. Abstracts of the papers follow below.

1. Professor Love's paper develops the elementary theory of the equation

$$\varphi(x) = \frac{f(x)}{x^a} + \lambda \int_a^\infty \frac{K(x, t)\varphi(t)}{x^a t^b} dt,$$

where $\varphi(x)$ is the unknown function, under the assumption that the functions $f(x)$ and $K(x, t)$ are bounded for $x \geq a$, $t \geq a$. This case possesses special interest on account of the fact that it can be solved by a direct extension of the method of Fredholm.

2. Dr. Borden considers with the equation

$$\Delta f'(x) + a(x)f'(x) + c(x)f(x) = 0$$

an adjoint equation. The invariants of the two equations under the group $f(x) = v(x)g(x)$ are compared, and also the invariants of the successive transforms of the two equations by the Laplace-Poisson transformations. (Poisson, *Journal de l'Ecole Polytechnique*, volume 6 (1806), pages 127-141.) Conditions are obtained for solutions of both equations in finite form and some remarks are made concerning the self-adjoint case.

3. The success of Einstein's theory of gravitation in removing the outstanding difference between theory and observation in the secular motion of the perihelion of Mercury, and its applicability to other problems of dynamical astronomy are used by Professor James as an argument in support of Bergson's thesis that there is no intrinsic order in nature and that the apparent mathematical order of the external universe has been imposed thereon by the intellect.

4. In this paper Professor Schmiedel contends that the first appearance of the idea of extension of the number concept is found in the Ahmes papyrus.

5. Maxime Bôcher (*Annals of Mathematics*, 1892) gave a geometrical proof of the theorem of Lucas (*Journal de l'Ecole Polytechnique*, 1879), that the roots of the derivative of a cubic are the foci of the maximum ellipse inscribed in the triangle, whose vertices are the roots of the cubic, and asked the question "Could not the roots of the derivative of the general polynomial be brought into connection with the foci of the higher plane curves?" Mr. Linfield answers the question by proving that the roots of the derivative of a polynomial of degree n are the foci of a curve of class $n - 1$ which touches the segments joining the roots of the polynomial, two and two, at their midpoints. He generalizes and proves that the roots of the derivative $f'(z)$ of the function $f(z) = \prod (z - z_r)^{\mu_r}$ are the foci of a curve of class $n - 1$ which touches the segment $z_r z_j$ at a point dividing it in the ratio $\mu_i : \mu_j$.

6. Professor Dodd finds that if the probability that the error of a measurement will lie between x and $x + dx$ is $c(1 + |hx|^{3/2})^{-1}dx$, with c and h constant, then the arithmetic mean of two measurements is the least probable value among all possible weighted means. Under the law $c_1(1 + h^2x^2)^{-1}$, however, the probability that any weighted mean, including the arithmetic mean, will differ from the true value by less than ϵ , is a constant, alike independent of the system of weights and the number of measurements. But under the laws $c_2[(1 + h^2x^2)(1 + k^2x^2)]^{-1}$, h and k equal or unequal, and $c_3(e^{hx} + e^{-hx})^{-1}$, the arithmetic mean is more reliable than any other weighted mean. The curves representing the foregoing laws of error closely resemble curves of the family $y = c_4e^{-h^2x^2}$, the probability function. The possibility of the arithmetic mean being a least probable value is associated with the fact that under certain laws of error the arithmetic mean of measurements is not as reliable as a single measurement. With an increase of the number of measurements, there is an increase in the probability that the larger errors will appear; and this may more than offset the neutralizing effect of errors of opposite sign.

7. In this note, Professor Lefschetz shows how the number of algebraically distinct real hypersurfaces belonging to real abelian varieties may be obtained from a certain period matrix which characterizes them. This is followed by the actual determination of the number in question for a large class of varieties defined by the possession of certain complex multiplications.

8. Professor Altshiller-Court's first paper appeared in full in the February BULLETIN.

9. The three vertices of a triangle and its orthocenter are the four vertices of an orthocentric quadrilateral having the property, pointed out by Carnot, that each vertex is the orthocenter of the triangle determined by the other three vertices. The four triangles thus formed, often referred to as an orthocentric group of triangles, have the same orthic triangle, and consequently the same nine-point circle, which may therefore be considered as belonging to the quadrilateral itself. Professor Altshiller-Court shows that the four circumcenters of an orthocentric group of triangles form another orthocentric quadrilateral, the two quadrilaterals having the same nine-point circle, the center of which is a center of symmetry of the two orthocentric groups. The four centroids of the given orthocentric group of triangles also form an orthocentric group, the two groups having the same nine-point center, which point is a center of similitude of the two orthocentric groups, the ratio of similitude being -3 .

10. It is well known that if an abelian group G has a subgroup of a given type, it has also a quotient group of this type and vice versa. If G is the abelian group of order p^m and of type $(m_1, m_2, \dots, m_\lambda)$ two of its subgroups H_1 and H_2 may be said to be of complementary types if each of them is simply isomorphic with the quotient group of G with respect to the other. A necessary and sufficient condition that two subgroups of types $(m_1', m_2', \dots, m_\lambda')$ and $(m_1'', m_2'', \dots, m_\lambda'')$ are complementary is that each of the following equations is satisfied: $m_1' + m_1'' = m_1, m_2' + m_2'' = m_2, \dots, m_\lambda' + m_\lambda'' = m_\lambda$. The number of the different types of subgroups which are complementary to the subgroups of a given type is equal to the number of the different sets of I -conjugate sub-

groups of G which are simply isomorphic with the subgroups of the given type, I -conjugate subgroups of G being subgroups which are conjugate under its group of isomorphisms. The main object of Professor Miller's paper is to establish some fundamental reciprocal properties of the sets of I -conjugate subgroups contained in certain types of abelian groups of order p^m . Among the theorems established is the following: If all the invariants of an abelian group G of order p^m are equal to each other, then the number of its subgroups of a given type is equal to the number of its subgroups which are simply isomorphic with the quotient group of G with respect to one of the former subgroups. In particular, the number of the distinct subgroups of order p^a contained in G is equal to the number of its distinct subgroups of index p^a .

11. It is known that two operators s_1, s_2 of order 3 whose product is of order 4 generate a group of infinite order unless these operators are otherwise restricted. In the present paper, Professor Miller considers the groups generated by s_1, s_2 when the conditions imposed on these operators are as follows: $s_1^3 = s_2^3 = (s_1 s_2)^4 = (s_1^2 s_2)^k = 1$, $k = 3, 4, 5$. When $k = 3$, these operators generate the group of order 48 obtained by extending the direct product of two cyclic groups of order 4 by means of an operator of order 3 which transforms this direct product into itself but is not commutative with any of its operators besides identity. When $k = 4$, the group generated by these operators is the simple group of order 168 and a new and useful definition of this important group is thus obtained. Finally, when $k = 5$, these operators generate a group of order 1,080 which is isomorphic with the simple group of order 360 with respect to its central. This group of order 1,080 does not contain the simple group of order 360 as a subgroup, but to each operator of the central quotient group there corresponds at least one operator of the group whose order is equal to the order of the corresponding operator in this quotient group.

12. In the present paper, Professor Rietz generalizes certain results of his first paper on this subject. In particular, the urn schema of the first paper that gives t/s for the correlation coefficient, where s is the number of balls in each of a pair of drawings, and t the number of balls taken at random

from the first drawing of a pair to be common to the pair, involves the hypothesis that the urn should be so maintained that the probability of drawing a white ball is $1/2$. This restriction is removed in the present paper so that the probability of drawing a white ball may be any constant p . The results are as simple as when $p = 1/2$, and the proofs are but slightly more complicated.

13. A classic problem in the mathematical theory of the flow of heat is to determine the temperature of a semi-infinite solid, initially at the temperature zero, whose plane face is kept at a temperature that varies in a known manner with the time. The solution usually given is that obtained by Duhamel and followed by Riemann. It treats the problem as a limiting case of a simpler one. In the method used by Professor Williams, a definite integral is assumed for a solution. It is suggested by the solution of a related problem on noticing an interchange of boundary conditions. The integrand and limits contain arbitrary elements. These are determined by substituting the assumed solution in the differential equation that it must satisfy.

14. In this note Professor Fort exhibits a series which includes both Dirichlet and factorial series as special cases. He proves a fundamental convergence theorem.

15. The recent study of boundary value problems has been confined to the case where the coefficients of the differential equation are continuous throughout the interval of definition. Except for Porter's and Bôcher's work on the solutions of a differential equation having regular singular points (Bôcher, "Oscillation theorems of Sturm," this BULLETIN, volume 5 (1898)), no attention has been given to the general theory of existence of solutions, oscillation theorems and expansions in normal functions for a system such as occurs in applied problems involving Legendre's, Bessel's or Lamé's differential equations. In this paper, Professor Ettlinger shows that a number of the theorems which hold for the continuous system go over for the system with regular singular points.

16. In a paper to appear in the June number of the *Annals of Mathematics*, Professor Ettlinger establishes the existence of

at least one real characteristic number for the system

$$\frac{d}{dx} \left[K(x, \lambda) \frac{du}{dx} \right] - G(x, \lambda) u = 0,$$

$$A_{i1}u(a) - A_{i2}K(a)u_x(a) - A_{i3}u(b) + A_{i4}K(b)u_x(b) = 0,$$

$$(i = 1, 2),$$

where the A_{ij} are constants and certain conditions are satisfied by K and G . This theorem is now extended to a system of the same form where $A_{ij}(\lambda)$ replaces the constant coefficients of the boundary conditions, subject to the additional restriction that $(A_{1i}A_{2j} - A_{2i}A_{1j})/(A_{1k}A_{2l} - A_{2k}A_{1l})$ decrease (or do not increase) as λ increases, for specified values of i, j, k, l .

17. Professor Evans's paper is an endeavor to connect by practical calculation the concepts of divergent series, integral equation, and Stieltjes integral. The equation treated is equation (1) in Lecture V of the author's Colloquium Lectures, when the kernel is of the form $f(x+s) + g(x-s)$, with f and g periodic with period ab as functions of a single argument. The solution is developed in Fourier series. An "imaginary" resolvent kernel is obtained in the form of a Fourier series, generally divergent and not even summable, which nevertheless yields with proper choice of the known function a summable series as solution of the equation, and thus affords a practical method of handling such equations. The equation treated, which is of the first kind, includes the equation of the second kind as a particular case.

18. In this paper, Mr. Steimley considers the general class of series of the form

$$Y(x) = c_0 + \sum_{n=1}^{\infty} c_n g(nx),$$

where $g(x)$ has the asymptotic character

$$g(x) \sim x^{P(x)} e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots \right),$$

which is valid for x approaching infinity in some sector V bounded by two rays extending from zero to infinity. $P(x)$

and $Q(x)$ are polynomials in x , a_i ($i = 1, 2, \dots$) are functions of the angle θ , which is the argument of x , varying in the sector V , c_i ($i = 0, 1, 2, \dots$) are independent of x . The character of the regions of convergence, of absolute convergence and of uniform convergence are determined. The character of the points on the boundary of the region of convergence is determined, except for an isolated set of points. The character of these isolated points is determined except in very special cases.

A necessary and sufficient condition that $Y(x)$ shall converge is determined. The uniqueness of expansion of functions in terms of $Y(x)$ is determined under special conditions.

19. In this paper, Professor Lunn shows that the general relations in thermodynamics can be arranged and the notations adjusted so as to show a complete duality analogous to the similar principles already known in geometry, mechanics and electromagnetism. Energy is dual to entropy and temperature to its own reciprocal, and a corresponding ranging in pairs occurs throughout the range of quantities characteristic of the thermodynamic properties of material systems.

As illustrations of the working of the principle various applications are suggested to the dual representation of physical relations and the dual interpretations of mathematical relations, to the attainment of symmetry of notation, to the transformation of the thermodynamic potentials, and to the criteria of equilibrium and stability.

20. In this paper Professor Borger proves, by using the Denjoy integral, that if the two functions $U(x, y)$, $V(x, y)$ of the real variables x, y are continuous in these variables jointly and possess finite partial derivatives of the first order, satisfying the Cauchy-Riemann differential equations at each point of the simply connected closed region R , then U and V are analytic functions of these variables.

An application to the theory of functions of a complex variable shows that if $W = U(x, y) + iV(x, y)$ and if U and V satisfy the conditions above referred to, then W' exists and is analytic.

21. The generalized Kummer surface is, for the genus three, a 3-way, M_3 , in S_7 obtained by equating the coordinates to

linearly independent theta functions of the second order and zero characteristic in three variables. Professor Coble shows that M_3 is the double manifold on a quartic locus Q_6 in S_7 . The fifteen coefficients of Q_6 , which are modular functions, satisfy a system of 63 cubic relations. On the other hand it is well known that seven points in a plane define a ternary quartic with isolated double tangents. Professor Coble has shown that the simplest linear system of invariants of degree three in the coordinates of each of the seven points has fifteen linearly independent members. These also satisfy a system of 63 cubic relations and are shown to be identical with the coefficients of Q_6 above. The modular group induces on these coefficients a group G of linear transformations. The invariants of G are invariants of the ternary quartic, which can be expressed either irrationally in terms of the seven points or transcendently in terms of the theta moduli.

22. By combining some theorems of Schottky with some results of his own, Professor Coble proves the following theorems concerning the Cayley symmetroid (a certain ten-nodal quartic surface with nine absolute constants): (1) The coordinates of the ten nodes of the symmetroid can be expressed uniformly by means of abelian modular functions of genus four. (2) Under integer linear transformation of the periods the symmetroid is algebraically transformed by *regular* Cremona transformation. (3) In particular those period transformations which are congruent to identity modulo 2 leave the symmetroid unaltered.

It thus appears that the ten nodes of the symmetroid are related to the abelian modular functions of genus four as the eight base points of a net of quadrics to the functions of genus three.

23. In this paper, Professor Wilczynski thinks of a congruence of lines as the totality of the generators of a one-parameter family of ruled surfaces, basing the analytic theory upon the previously developed theory of ruled surfaces, and connecting this theory with the theory of congruences by a chain of several related theories which seem to be entirely new. The principal advantage gained for the theory of congruences consists in the new expressions found for the invariants, expressions which are elegant in form and directly

applicable to a congruence whose developables are not known. Moreover the auxiliary quantities in terms of which these invariants are expressed are interpretable in terms of the new theories of one-parameter families of ruled surfaces, and nets of ruled surfaces.

24. According to Lüroth (*Crelle's Journal*, volume 68, pages 185-190) the analytic method of investigating problems of the sort here considered, as outlined by him and suggested by a paper of Clebsch, is too difficult on account of the complicated relations between the coefficients of the given and required elements and forms. Making use of a purely geometric method, Lüroth solves a number of problems in which the curves are conics.

Professor Emch shows that these and more general problems of this type may be solved by a relatively simple analytical method which consists chiefly in the utilization of a singular null system of the following type:

$$\begin{aligned}
 \rho x_1 &= 0 & + \alpha_2 p_{12} + \alpha_3 p_{13} + \alpha_4 p_{14}, \\
 \rho x_2 &= -\alpha_1 p_{12} + 0 & + \alpha_3 p_{23} + \alpha_4 p_{24}, \\
 \rho x_3 &= -\alpha_1 p_{13} - \alpha_2 p_{23} + 0 & + \alpha_4 p_{34}, \\
 \rho x_4 &= -\alpha_1 p_{14} - \alpha_2 p_{24} - \alpha_3 p_{34} + 0.
 \end{aligned}
 \tag{1}$$

For $i = 1, 2, 3, 4$; $k = 1, 2, 3, 4$; $i \neq k$, the x_i are the coordinates of a point, the α_i those of a plane, the p_{ik} those of a straight line. The determinant of the transformation (null system = involuntary correlation throughout) is

$$(p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23})^2 = 0.$$

This null system is therefore singular and has the following geometric meaning. The plane (α_i) cuts the straight line (p_{ik}) in a point (x_i) whose coordinates are precisely those given by (1). The method based upon this system is believed to be new.

25. Professor Boutroux considers the following fundamental problem in the study of multiform functions satisfying differential equations of the first order. Calling $f(x, C)$ the general solution of an equation, find the values $C_1, C_2, \dots$ of C which belong to the same function (that is, are such that $f(x, C_1)$,

etc., are branches of one and the same multiform function). The substitutions exchanging these values $C_1, C_2, \dots$, form a discontinuous group G , and to investigate that group, we have to consider certain functions $\psi(C)$ (substituting functions). As a rule these last functions are themselves multiform functions. But some families of differential equations have the property that to build up the corresponding group G , it is not necessary to use the whole functions $\psi(C)$ but merely special branches or uniform "elements" of these functions which can be defined simply. Thus the difficulties arising from the multiform character of the functions in the problem are overcome.

As an example, one may take the functions $z(x)$ satisfying $z(dx/dx) = mz + 2(x^2 - 1)$. Choosing for C a parameter which determines z at $x = \infty$, one first sees that all the substitutions of G are of the form $\left[C, C + m \int_{\Gamma} z dx \right]$, where Γ is any closed path starting from $x = \infty$. Now it is possible to find two special substituting functions of this type, ψ_1, ψ_2 , each of which is holomorphic in the whole C -plane limited by two straight cuts (straight lines joining two certain points to $x = \infty$). Considering the said functions exclusively in these limited fields, one can define uniformly for all values of C , two substitutions $[C, \psi_1(C)]$ and $[C, \psi_2(C)]$. These particular substitutions, combined with a third one of the form $[C, C + \text{constant}]$ will be sufficient, if properly chosen, to build up the total group G .

26. The coefficient of heat conductivity of most fluids, determined from observations made under controlled conditions, in which motion is eliminated, is very small. Moreover, in order to apply the results of the classical theory of the flow of heat to fluids, they must be regarded as solids. In this paper, Dr. McEwen presents the derivation of differential equations of the flow of heat in a fluid due to a convective, or alternating circulation of small fluid volumes, both when the average fluid velocity across a given surface is zero, and when it has an appreciable value. Two sets of equations are derived; one for the relatively cold elements, the other for the warm elements. The temperature that would be indicated by a thermometer depends upon both the warm and cold elements since they are intermixed. The measurable tempera-

ture is therefore a mean of these two, and a proper combination of the first two sets of equations results in a single set for determining this temperature.

27. The usual method of deducing frequency curves from assumptions as to the manner in which elementary errors or deviations arise often requires a larger number of measurements than are available in practical applications. Moreover, the difficulty of determining and handling the proper frequency curve except in special cases in which the Gaussian, or normal law holds, restricts the use of such deductive methods, and often results in erroneous applications of the simple normal law. By arranging the observations in their order of magnitude, and regarding two groups, those above the average and those below, as samples of corresponding groups of an indefinitely large sample, Dr. McEwen reduces the problem of the precision of averages to the simple one of variation in proportions. Regarding the observed average of the whole series as the best estimate of that of an indefinitely large sample, and changing the proportion of observations above and below the average, the probability of these variations in proportions corresponds to that of the resulting variation in the average computed from the new group values and the observed constant proportion. A frequency distribution of averages thus computed gives directly the probability that the true average falls within given limits. From similar determinations of the frequency distribution of the averages of two sets of observations, the probability can be found that the true values differ by an amount exceeding any specified value. A modification of this "group proportion" method not requiring the determination of the frequency distributions serves to estimate rapidly the probability that an observed difference between two averages arose by chance. The above results apply to any number of observations, and to any type of frequency distribution of the observations.

O. D. KELLOGG, *Secretary of the Southwestern Section.*

ARNOLD DRESDEN, *Secretary of the Chicago Section.*

PONCELET POLYGONS IN HIGHER SPACE.

PROFESSOR ALBERT A. BENNETT.

(Read before the American Mathematical Society December 30, 1919.)

LET there be given a linear projective space of $2n$ dimensions. A point of the space may be denoted by P and its dual figure by P' . Thus a P' is a linear space of $2n - 1$ dimensions.

The totality of P 's in the space is infinity to the order $2n$, and the totality of P' 's is of course of this same order. We shall select from these totalities a Q_n and a Q_n' respectively, general quadratic loci of infinity to the order n of elements, where Q_n consists of P 's, and Q_n' of P' 's.

For Q_n and Q_n' not in specialized relation to each other we have a two-two correspondence of the following form: Each P of Q_n meets two P' 's of Q_n' , and each P' of Q_n' meets two P 's of Q_n . Starting with any point of Q_n , a succession of points of Q_n is determined, where furthermore consecutive points of the sequence may be joined by lines. The succession of lines forms then a single "broken line" as this term is used in projective geometry. It may or may not happen that the broken line closes into a polygon. Except for degenerate cases corresponding to coincident P 's or P' 's, and it being supposed that Q_n and Q_n' are not degenerate, it may be proved that the closure of the broken line is determined by the relative positions of Q_n and Q_n' and is independent of the element selected as initial.

This may be called a theorem of Poncelet polygons in higher spaces. For $n = 1$, the theorem is the usual one.

It should be emphasized that the case for $n > 1$ is not the logical equivalent of the case for $n = 1$, since there are n independent parameters in any case. The proof of the theorem is immediate by reference to general theorems on algebraic correspondences or to theta functions, the quadratics Q_n and Q_n' determining theta functions of genus n , and affording one of the simplest illustrations of their character.

A second generalization and one which applies to three-space is to spaces of $2n - 1$ dimensions generally, $n > 1$, the P , P' , Q_n , Q_n' being as above. Any P' of Q_n' may be viewed

as a $(2n - 1)$ -space tangent to the n -dimensional quadratic cone K_n' of $(n - 2)$ -spaces also represented as Q_n' . While a P' of Q_n' meets Q_n in a conic, the two $(2n - 2)$ -spaces, L' , tangent to K_n' and contained in P' , which are also tangent to Q_n , determine two points of tangency on Q_n . This correspondence is again two-two, and for it the same theorem holds. The case $n = 2$ leads to the study of Kummer's surface and the theorem is in substance familiar in this case. Cf. Hudson, *Kummer's Quartic Surface*, Cambridge, 1905, page 196, and Zeuthen, *Lehrbuch der abzählenden Methoden der Geometrie*, Leipzig, 1914, page 276.

WASHINGTON, D. C.,
November, 1919.

ON THE RECTIFIABILITY OF A TWISTED CUBIC.

BY DR. MARY F. CURTIS.

(Read before the American Mathematical Society December 31, 1919.)

If the space curve

$$(1) \quad x_i = a_i t^n + b_i t^{n-1} + \dots + k_i t + l_i \quad (i = 1, 2, 3)$$

is a helix, it is algebraically rectifiable. For if it is a helix, it makes with a fixed direction a constant angle and $\sqrt{x'|x'} = (x'|\alpha)^*$, where $\alpha_1, \alpha_2, \alpha_3$ are constants, not all zero; then the arc

$$(2) \quad s = \int_{t_0}^t \sqrt{x'|x'} dt$$

is an integral rational function of t , not identically zero, and the curve (1) is algebraically rectifiable.

It is not, however, in general true, that if (1) is algebraically rectifiable, it is a helix. It will be true, provided (2) is an algebraic function only when $(x'|x')$ is a perfect square of the form $(x'|\alpha)^2$. This condition is fulfilled in the case of the twisted cubic:

$$(3) \quad x_1 = at, \quad x_2 = bt^2, \quad x_3 = ct^3, \quad abc \neq 0,$$

* If $a : (a_1, a_2, a_3)$ and $b : (b_1, b_2, b_3)$ are two triples, then by $(a|b)$ we mean their inner product: $a_1 b_1 + a_2 b_2 + a_3 b_3$.

and hence, as I pointed out in a note on page 87 of volume 25 (1918) of this BULLETIN, the cubic (3) is algebraically rectifiable when and only when it is a helix.

As a generalization of this result Professor Tsuruichi Hayashi, in the November number of the current volume of the BULLETIN, has for the twisted cubic

$$(4) \quad x_i = a_i t^3 + b_i t^2 + c_i t + d_i \quad (i = 1, 2, 3, \quad |abc| \neq 0),$$

which, by a change of parameter and axes, he writes in the form

$$(5) \quad x_1 = a_1 t^3 + c_1 t, \quad x_2 = a_2 t^3 + b_2 t^2, \quad x_3 = a_3 t^3, \quad a_3 b_2 c_1 \neq 0,$$

the theorem: The conditions

$$(6) \quad a_2 = 0, \quad 9a_3^2 c_1^2 = 12a_1 c_1 b_2^2 + 4b_2^4,$$

are necessary and sufficient both that (5) be a helix and that (5) be algebraically rectifiable.

This theorem, as stated, is not true. The cubic (5) may be algebraically rectifiable if not a helix, for its arc (2), where

$$(x'|x') = 9(a|a)t^4 + 12a_2 b_2 t^3 + (6a_1 c_1 + 4b_2^2)t^2 + c_1^2,$$

is an algebraic function, not only when $(x'|x')$ is a perfect square but also when $(x'|x')$ is a cubic with a multiple root, namely, when

$$(7) \quad a_2 \neq 0, \quad (a|a) = 0, \quad 2(3a_1 c_1 + 2b_2^2)^2 + 243a_2^2 b_2^2 c_1^2 = 0.$$

Hence Professor Hayashi's theorem is true for the cubics (5) only if those for which (7) holds are excluded.

Since the cubics (5) for which (7) holds are imaginary, we may say: *A real twisted cubic (5) is algebraically rectifiable, when and only when it is a helix.*

It is interesting to note that the facts for the cubics given by (5) do not quite correspond to those for the cubics given by (4), from which (5) was deduced, because, in the reduction of (4) to (5) certain imaginary cubics given by (4) have been excluded; among them are the helices, for which $(x'|x')$ for (4) reduces to the form: $(kt^2 + lt)^2 \neq 0$, and the algebraically rectifiable cubics, not helices, obtained when $(x'|x')$ for (4) is of the form: $mt + n$, $m \neq 0$. But, as these curves are all imaginary, the theorem just stated for the cubics (5) holds equally well for the cubics (4).

In order that the general twisted cubic (4) be algebraically rectifiable, it is sufficient, but not necessary, that it be a helix. By demanding algebraic rectifiability of a particular type, we may obtain a condition in terms of the curvature and torsion of (4), which is both necessary and sufficient:

The general twisted cubic (4) may be represented by equations of the form (4), where t is its arc, when and only when it is of constant curvature and torsion.

For a necessary and sufficient condition that (4) may be so represented is that $(x'|x')$ is a constant, not zero, and this in turn can be shown to be necessary and sufficient that (4) is a curve of constant curvature and torsion.

WELLESLEY COLLEGE,
November 15, 1919.

NOTE ON LINEAR DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER WHOSE SOLUTIONS SATISFY A HOMOGENEOUS QUADRATIC IDENTITY.

BY DR. C. N. REYNOLDS, JR.

(Read before the American Mathematical Society December 31, 1919.)

In this paper I shall prove that if a given homogeneous linear differential equation of the fourth order has four linearly independent real solutions which satisfy a non-singular homogeneous quadratic identity, then it may be reduced to a form which may be said to be self-adjoint with respect to the third row of the wronskian of any four linearly independent solutions. I shall then evaluate the signature of the quadratic identity in terms of the coefficients of the reduced equation and derive several theorems concerning the zeros of solutions of such equations.

Brioschi* has shown that if $y_1(x)$, $y_2(x)$, $y_3(x)$ and $y_4(x)$ are four linearly independent real solutions of the equation

$$(1) \quad p_0(x)y^{(4)}(x) + 4p_1(x)y'''(x) + 6p_2(x)y''(x) + 4p_3(x)y'(x) + p_4(x)y(x) = 0 \quad (p_0(x) \neq 0),$$

* "Les invariants des équations différentielles linéaires," *Acta Mathematica* (1890), vol. 14, p. 233.

with real coefficients which together with a sufficient number of their derivatives are continuous, and if

$$(2) \quad \sum_{i=1}^4 \sum_{j=1}^4 A_{ij} y_i(x) y_j(x) \equiv 0 \quad (A_{ij} = A_{ji}),$$

where not all of the A 's are zero; then, by means of a suitable change of variables we can reduce our equations (1) and (2) to the form

$$(3) \quad \begin{aligned} &v''(z) + 6q_2(z)v''(z) + 9q_2'(z)v'(z) \\ &+ [3q_2''(z) + C]v(z) = 0 \quad (z = f(x), \quad f'(x) \neq 0), \end{aligned}$$

$$(4) \quad \sum_{i=1}^4 \sum_{j=1}^4 A_{ij} v_i(z) v_j(z) \equiv 0 \quad (A_{ij} = A_{ji}),$$

where

$$\Delta \equiv \begin{vmatrix} v_1(z), & v_2(z), & v_3(z), & v_4(z) \\ v_1'(z), & v_2'(z), & v_3'(z), & v_4'(z) \\ v_1''(z), & v_2''(z), & v_3''(z), & v_4''(z) \\ v_1'''(z), & v_2'''(z), & v_3'''(z), & v_4'''(z) \end{vmatrix} \equiv (v_1, v_2', v_3'', v_4''')$$

is a constant different from zero. Cels* has called the equation satisfied by the cofactors of $v_i''(z)$ ($i = 1, 2, 3, 4$) in Δ the "equation adjoint to (3) with respect to the third row of Δ ." By repeated differentiation of (v_1, v_2', v_3'') we find that it satisfies (3). This equation may therefore be said to be self-adjoint with respect to the third row of Δ .

If we now define $v_i(z)$ by means of the boundary conditions

$$\begin{aligned} v_i^{(i-1)}(\zeta) &= 1, & (i, k = 1, 2, 3, 4; \\ v_i^{(k-1)}(\zeta) &= 0, & k \neq i) \end{aligned}$$

then $v(z)$, any solution of (3), may be expressed in the form

$$v(z) = v(\zeta)v_1(z) + v'(\zeta)v_2(z) + v''(\zeta)v_3(z) + v'''(\zeta)v_4(z).$$

In particular, we can so express the cofactors of $v_i''(z)$ in Δ :

$$\begin{aligned} (v_2, v_3', v_4''') &= v_3, \\ -(v_1, v_3', v_4''') &= -v_2 & + 6q_2(\zeta)v_4, \\ (v_1, v_2', v_4''') &= v_1 & - 6q_2(\zeta)v_3 & - 3q_2'(\zeta)v_4, \\ -(v_1, v_2', v_3''') &= 6q_2(\zeta)v_2 - 3q_2'(\zeta)v_3 - [36q_2^2(\zeta) - C]v_4. \end{aligned}$$

* "Sur les équations différentielles linéaires ordinaires," *Annales Scient. de l'Ecole Normale Supérieure* (1891), vol. 8, 3d ser., p. 341.

Substituting these values of our three rowed determinants in the expansion of the determinant

$$(v_1, v_2', v_3, v_4''') \equiv 0$$

in terms of the elements of its third row, we have

$$\begin{aligned} & [v_1 + (-3q_2(\zeta) + \tfrac{1}{2})v_3 - 3q_2'(\zeta)v_4]^2 \\ & - [v_1 + (-3q_2(\zeta) - \tfrac{1}{2})v_3 - 3q_2'(\zeta)v_4]^2 \\ & - [v_2 - 6q_2(\zeta)v_4]^2 + Cv_4^2 \equiv 0. \end{aligned}$$

Therefore, the signature of the quadratic form in (4) is zero if C is positive, two if C is negative.

If $v_1(z)$ and $v_2(z)$ are any solutions of (3) then

$$\begin{aligned} S(v_1, v_2) & \equiv (v_1''' + 6q_2v_1' + 3q_2'v_1)(v_2''' + 6q_2v_2' + 3q_2'v_2) \\ & + C(v_1''v_2 - v_1'v_2' + v_1v_2'' + 6q_2v_1v_2) \end{aligned}$$

is constant since

$$\frac{d}{dz} S(v_1, v_2) \equiv 0.$$

If $v(\zeta) = v'(\zeta) = 0$ then $S(v, v) = v''^2(\zeta) \geq 0$. Therefore, no solution $v(z)$ of (3) for which $S(v, v) < 0$ can have a multiple zero. Similarly we find that

If $C < 0$ and $S(v, v) < 0$ then $v(z)$ has no zeros.

If $C[S(v_1, v_1)S(v_2, v_2) - S^2(v_1, v_2)] > 0$ then $v_1(z)$ and $v_2(z)$ have no zeros in common.

Brioschi has shown that if $C > 0$ then any solution of (3) is a bilinear form in the solutions of

$$(5) \quad u''(z) + [\tfrac{3}{2}q_2(z) + r]u(z) = 0, \quad 4r^2 = C, \quad r > 0,$$

$$(6) \quad w''(z) + [\tfrac{3}{2}q_2(z) - r]w(z) = 0.$$

If $u_1(z)$, $u_2(z)$, $w_1(z)$ and $w_2(z)$ are defined by the boundary conditions:

$$\begin{aligned} u_i^{(i-1)}(\zeta) &= 1, & w_i^{(i-1)}(\zeta) &= 1, & (i, k &= 1, 2; \\ u_i^{(k-1)}(\zeta) &= 0, & w_i^{(k-1)}(\zeta) &= 0, & k &\neq i) \end{aligned}$$

then $v(z)$, any solution of (3), may be written in the form

$$v(z) = v(\zeta)u_1(z)w_1(z)$$

$$\begin{aligned}
& -\frac{1}{4r}[v''' + (6q_2 - 2r)v' + 3q_2'v]_{z=\zeta}u_1(z)w_2(z) \\
& + \frac{1}{4r}[v''' + (6q_2 + 2r)v' + 3q_2'v]_{z=\zeta}u_2(z)w_1(z) \\
& + \frac{1}{2}[v''(\zeta) + 3q_2(\zeta)v(\zeta)]u_2(z)w_2(z).
\end{aligned}$$

$S(v, v) = 0$ is therefore the necessary and sufficient condition that $v(z)$ be factorable as the product of a solution of (5) by a solution of (6). Therefore, if $S(v, v) = 0$ the number of zeros of $v(z)$ in a given closed interval (a, b) will be limited by Sturm's theorems concerning the zeros of solutions of (5) and (6). If, for example, the equations (5) and (6) have k_1 and k_2 for their respective indices of oscillation in (a, b) then every solution of (5) [(6)] will have k_1 or $k_1 + 1$ [k_2 or $k_2 + 1$] zeros in the interval (a, b) , and every solution, $v(z)$, of (3) for which $S(v, v) = 0$ will have $k_1 + k_2$, $k_1 + k_2 + 1$, or $k_1 + k_2 + 2$ zeros in (a, b) . If, in particular, $q_2(z) < 0$ and $0 < C < 9q_2^2(z)$ throughout (a, b) then the coefficients of $u(z)$ and $w(z)$ in (5) and (6) will be negative throughout (a, b) and $k_1 = k_2 = 0$.* Therefore, if $q_2(z) < 0$ and $0 < C < 9q_2^2(z)$ throughout (a, b) then no solution of (3) for which $S(v, v) = 0$ has more than two zeros in (a, b) , and some such solutions have no zeros in (a, b) .

WESLEYAN UNIVERSITY,
MIDDLETOWN, CONN.

AN ACKNOWLEDGMENT OF PRIORITY.

BY PROFESSOR A. A. BENNETT.

IN the July, 1919, number of the BULLETIN (volume 25, page 455), I gave a solution condensed to about two pages, of a problem proposed by the Philosophical Faculty of the University of Berlin as a prize problem. My attention has been called to the fact that this question is completely handled in an analogous manner by Harold Hilton, in his book on "Homogeneous Linear Substitutions," 1914, in Chapter VI, section 8. Similar results are given in the same chapter for

* See Bôcher's Méthodes de Sturm, Paris, 1917, p. 52.

symmetric and Hermitian matrices. This book, while familiar to the American public, was not available to me at the time, and I did not realize that the problem proposed had been completely settled previous to the occurrence of the war.

February 9, 1920.

DICKSON'S HISTORY OF THE THEORY OF NUMBERS.

ON page 130 of my review of Dickson's History I speak of two pages of titles at the end of Chapter XII as "not reported on." This is an error. With the exception of certain ones marked with an asterisk which give papers not obtainable by the author the content of the papers is sufficiently indicated, the papers not being of importance to warrant more detailed report.

The phrase "list of references," on the top of page 132, is perhaps misleading. Of course the book is in no sense to be compared with the useless lists of titles of papers which the compiler may or may not have glanced at. My intention in the paragraph was to bring out the distinction which modern historians seem to make between a list of events and the relations and connections between events.

D. N. LEHMER.

NOTES.

THE December number (volume 21, number 2) of the *Annals of Mathematics* contains the following papers: "A property of cyclotomic integers and its relation to Fermat's last theorem," by H. S. VANDIVER; "Surfaces of rotation in a space of four dimensions," by C. L. E. MOORE; "The circle nearest to n given points, and the point nearest to n given circles," by J. L. COOLIDGE; "Singular solutions of differential equations of the second order," by E. M. COON and R. L. GORDON; "Note on a class of integral equations of the second kind," by C. E. LOVE; "Concerning sense on closed curves in

non-metrical plane analysis situs," by J. R. KLINE; "On the theory of summability," by GLENN JAMES; "On the consistency and equivalence of certain generalized definitions of the limit of a function of a continuous variable," by L. L. SILVERMAN.

THE Mathematical association of Japan for secondary education issued the first number of volume 1 of its *Journal* under date of April, 1919. It is published by M. KABA, of Tokyo, as chief editor, assisted by M. KURODA, M. WATANABE, S. NAGAO and H. FURUKAWA. Professor T. HAYASHI is president of the Association.

It is announced that a new quarterly devoted to the mathematics of finance, the *Giornale di Matematica Finanziaria*, will be published under the auspices of the higher institutes of commerce and the chambers of commerce of Turin and Genoa. The business offices are at 73, Corso Vittorio Emanuele, Turin.

THE Paris Academy of sciences has awarded its Bordin prize for 1919 to Professor S. LEFSCHETZ, of the University of Kansas, for his memoir "On certain invariant numbers of algebraic varieties, with application to abelian varieties." The Academy announces the following subject for its Grand prize, to be awarded in 1922: The determination of extensive classes of surfaces by given properties of their geodesics considered in ordinary space.

PROFESSOR C. SERVAIS, of the University of Ghent, has been elected an active member, and Professor T. DE DONDER, of the University of Brussels, a corresponding member of the Belgian academy of sciences.

PROFESSORS J. PLEMELJ, of the University of Czernovitch, and R. SUPPANTSCHITSCH, of the technical school of Vienna, have been appointed professors of mathematics at the re-organized University of Lioubliana (Laibach), Jugoslavia.

MR. J. H. JEANS, of Cambridge, formerly professor of mathematics at Princeton University, has been nominated as secretary of the Royal society of London. Professor G. N.

WATSON, of the University of Birmingham, has been elected a fellow of the society.

SIR J. J. THOMSON, master of Trinity College, Cambridge, who recently resigned the Cavendish professorship of experimental physics, has been elected to the newly established professorship of physics.

CAPT. L. L. BURCHNALL, scholar of Christ Church, Oxford, has been appointed lecturer in mathematics at the University of Durham.

DR. R. J. T. BELL, of the University of Glasgow, has been appointed to the chair of pure and applied mathematics in the University of Otago, New Zealand.

PROFESSORS N. B. MACLEAN and N. R. WILSON have returned to the University of Manitoba, from active service in France.

PROFESSOR E. W. BROWN delivered a lecture on the history of mathematics before the Gamma Alpha fraternity of Yale University on February 26. This is the first of a series on the history of the several sciences. The Astronomical Society of the Pacific has awarded its Bruce Medal for 1920 to Professor BROWN.

AT Yale University, assistant professor W. R. LONGLEY has been promoted to a professorship and Mr. J. K. WHITEMORE to an assistant professorship of mathematics.

PROFESSOR P. P. BOYD, of the University of Kentucky, has been elected president of the Kentucky academy of sciences.

BOSTON University has concluded an arrangement for the exchange of professorships in mathematics for the college year 1920-1921 with Tsing Hua College, Peking, China. Professor R. E. BRUCE, chairman of the department in Boston University, will exchange with Professor A. H. HEINZ, of Tsing Hua.

ASSISTANT professor K. P. WILLIAMS, of Indiana University, has returned from military service and been promoted to an associate professorship of mathematics.

DR. C. C. CAMP, of the University of Illinois, has been appointed assistant professor of mathematics at Iowa State College.

ASSISTANT professor ARTHUR RAMSEY has been promoted to a professorship of mathematics at Grove City College.

MISS OTTILIA W. DUEKER has been appointed professor of mathematics and dean of women at the Friends' University, Wichita, Kansas.

DR. J. S. MILLER, of Emory and Henry College, has been appointed professor of mathematics at Hampden-Sidney College.

MR. J. E. DOTTERER has been appointed professor of mathematics and physics at Manchester (Indiana) College.

ASSISTANT professor C. P. SOUSLEY, of Pennsylvania State College, has been promoted to an associate professorship of mathematics.

MR. V. G. GROVE, of Cornell University, has been appointed assistant professor of mathematics at the Michigan Agricultural College.

AT Cornell University, the following appointments to instructorships in mathematics have been made: Dr. H. C. M. MORSE, of Harvard University, Mr. P. A. FRALEIGH, of Dartmouth College, and Mr. F. M. LUFKIN.

DR. W. H. WILSON, of the Massachusetts Institute of Technology, has been appointed instructor in mathematics at the University of Iowa.

THE following instructors in mathematics have been appointed at the University of Wisconsin: C. D. EHRLMAN, I. W. TRAYLER, J. E. DAVIS, M. INGRAHAM, F. E. RECHART, Miss E. M. F. FELTGES, and Miss B. ALBRIGHT.

PROFESSOR H. G. ZEUTHEN, of the University of Copenhagen, died January 6, 1920, at the age of eighty years.

He spent the summer of 1919 in writing a seventy-page paper on the origin of algebra.

REV. C. J. BORGMAYER, professor of mathematics and astronomy at St. Louis University, died December 6, 1919, at the age of fifty-eight years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- BARTHEL (E.). *Architektonik der Kegelschnitte*. Autographiert. Strassburg, Selbstverlag, 1916.
- BAUDET (P. J. H.). *Groepentheoretische onderzoekingen*. S. Gravenhage, Nijhoff, 1918. 8vo. 114 pp.
- BECK (H.). *Koordinaten-Geometrie*. Berlin, Springer, 1919. 8vo. 432 pp. Geb. M. 31.00
- BLASCHKE (W.). *Kreis und Kugel*. Leipzig, Veit, 1916. 10 + 169 pp. M. 6.50
- BOREL (E.). See PETROVITCH (M.).
- BOWLEY (A. L.). *General course of pure mathematics from indices to solid analytical geometry*. Oxford, University Press, 1919. Cr. 8vo. 7s. 6d.
- CRAMER (H.). *Sur une classe de séries de Dirichlet*. (Diss.) Upsala, 1917.
- D'Ailly (G. H.). *Sur quelques propriétés des coefficients de certaines séries de Laurent*. (Diss.) Upsala, 1917.
- DERKSEN (H. A.). *Inleidning tot de hoogere wiskunde, behandelende grafische voorstellingen en de beginselen der differentiaal- en integraalrekening*. Zutphen, Thieme, 1919. 8vo. 165 pp.
- DÜSING (K.). *Die Elemente der Differential- und Integralrechnung in geometrischer Methode*. Mit zahlreichen Beispielen aus der technischen Mechanik von E. Preger. 4te Auflage. Leipzig, M. Jancke, 1917.
- FRAENKEL (A.). *Einleitung in die Mengenlehre*. Berlin, Springer, 1919. 8vo. 156 pp. M. 10.00
- HURWITZ (A.). *Vorlesungen über die Zahlentheorie der Quaternionen*. Berlin, Springer, 1919. 8vo. 74 pp. M. 8.00
- ISENKRANZ (C.). *Das Endliche und das Unendliche*. Münster i. W., H. Schöningh, 1915. Gr. 8vo. 8 + 332 pp. M. 4.80
- KOMMERELL (K.). See KOMMERELL (V.).
- KOMMERELL (V.) und KOMMERELL (K.). *Analytische Geometrie*. I. 3te Auflage. Tübingen, Laupp, 1917. Geh. M. 3.60

- LINDEMANN (C.). Konjugierte Punkte bei Widerstandsaufgaben der Variationsrechnung. (Diss.) Breslau, 1917.
- MATEMATISK FORENING. Nyere Undersøgelser over Integralregningen. Holdt 1. København, Gjellerup, 1917.
- MERTENS (F.). Ueber die Bildung zyklischer Gleichungen in einem gegebenen Rationalitätsbereich. Wien, 1916. M. 2.25
- PAPERS set in the mathematical tripos. Part 1. 1913-1917. Cambridge. University Press, 1919. Cr. 8vo. 4 + 82 pp. 3s.
- PETROVITCH (M.). Les spectres numériques. Préface de M. Emile Borel. Paris, Gauthier-Villars, 1919. 8vo. 8 + 110 pp.
- PORTZEHL (W.). Zur Methode der orientierten Elemente in mehrdimensionalen Räumen. (Diss.) Königsberg i. Pr., 1917.
- PREGER (E.). See DÜSING (K.).
- RIEMANN (B.). Ueber die Hypothesen, welche der Geometrie zu Grunde liegen. Neu herausgegeben von H. Weyl. Berlin, Springer, 1919. 8vo. 47 pp. M. 5.60
- RUTGERS (J. G.). See SCHUH (F.).
- SCHUH (F.) en RUTGERS (J. G.). Compendium der hoogere wiskunde. Derde deel. Differentiaalrekening. Analytische meetkunde van de ruimte. Integraalrekening. Groningen, Noordhoff, 1919. 8vo. 239 pp. Fl. 7.50
- STIELTJES (T. J.). Oeuvres complètes, publiées par les soins de la Société mathématique d'Amsterdam. Tome II. Groningen, Noordhoff, 1918. 4 + 603 pp.
- THOMMECK (B.). Ueber einen mechanischen Apparat zur Bestimmung der Nullstellen ganzer rationaler Funktionen mit komplexen Koeffizienten. (Diss.) Bonn, 1917.
- WEYL (H.). See RIEMANN (B.).

II. ELEMENTARY MATHEMATICS.

- COMBEROUSSE (C. DE). Cours de mathématiques. Tome I, 2e partie: Algèbre élémentaire. 6e édition, entièrement refondue et augmentée, par R. de Montessus de Ballore. Paris, Gauthier-Villars, 1919. 8vo. 500 pp. Fr. 11.25
- DAVISON (C.). Plane trigonometry for secondary schools. Cambridge, University Press, 1919. Cr. 8vo. 8 + 334 pp. 6s. 6d.
- HEFFTER (L.). Ebene und sphärische Trigonometrie. 3te Auflage. (Sammlung Götschen, Nr. 99.) Berlin, Götschen, 1917. M. 1.00
- LIEZMANN (W.). Aufgabensammlung und Leitfaden der Geometrie. Ausgabe A, für Gymnasien. Unterstufe. Leipzig, Teubner, 1917. M. 2.80
- . Leitfaden der Mathematik. Ausgabe A, für Gymnasien. Unterstufe. Leipzig, Teubner, 1917. M. 2.20
- LÖFFLER (E.). See LÖRCHER (O.).
- LÖRCHER (O.) und LÖFFLER (E.). Methodischer Leitfaden der Geometrie nebst einer Vorschule der Trigonometrie für höhere Lehranstalten. 3te Auflage. Stuttgart, Grub, 1917. 12 + 205 pp. M. 2.80

- MOČNIK (—.) und SPIELMANN (—.). Anfangsgründe der Geometrie für die 1te bis 3te Klasse der Mittelschulen. 29te geänderte Auflage. Wien, Tempaky, 1916. 125 pp. Kr. 1.80
- MOLESWORTH (G.). Decimal tables. London, Spon, 1919. 16mo. 7 + 118 pp.
- MONTESUS DE BALLORE (R. DE). See COMBEROUSSE (C. DE).
- MORGAN (F. M.). See YOUNG (J. W.).
- MÜLLER (C. H.). Lehr- und Uebungsbuch der Geometrie. 2ter Teil. Ausgabe B, für die Oberstufe der Gymnasien. 3te Auflage. Leipzig, Freytag, 1916.
- PALMER (C. I.). Practical mathematics for home study, being the essentials of arithmetic, geometry, algebra and trigonometry. New York, McGraw-Hill, 1919. 8vo. 23 + 493 pp.
- SPIELMANN (—.). See MOČNIK (—.).
- YOUNG (J. W.) and MORGAN (F. M.). Plane trigonometry and numerical computation. New York, Macmillan, 1919. 7 + 122 pp. \$1.25

III. APPLIED MATHEMATICS.

- APPELL (P.). Traité de mécanique rationnelle. Tome I: Statique. Dynamique du point. 4e édition, entièrement refondue. Paris, Gauthier-Villars, 1919. 8vo. 8 + 620 pp. Fr. 54.00
- BALU (E.). Géodésie topométrique. 3e fascicule: Détermination du point de relèvement. Paris, Gauthier-Villars, 1919. 8vo. 6 + 58 pp. Fr. 6.00
- BLOCH (O.). Die Ortskurven der graphischen Wechselstromtechnik, nach einheitlicher Methode bearbeitet. Zürich, Rascher, 1917. M. 9.00
- CONNAISSANCE des temps ou des mouvements célestes à l'usage des astronomes et des navigateurs, pour l'an 1921, publiée par le Bureau des Longitudes. Paris, Gauthier-Villars, 1919. 8vo. 30 + 752 pp. Fr. 9.00
- CROWTHER (J. A.). A manual of physics. London, Henry Frowde, 1919. 16s.
- DEMOZAY (L.). Relations remarquables entre les éléments du système solaire. Paris, Gauthier-Villars, 1919. 8vo. 142 pp. Fr. 12.00
- EGGERT (O.). See JORDAN (W.).
- EPHÉMÉRIDES nautiques pour l'an 1920, publiées par le Bureau des Longitudes. Extrait de la Connaissance des temps, à l'usage des marins. Paris, Gauthier-Villars, 1919. 8vo. 138 pp. Fr. 4.50
- GEIGER (H.). See MAKOWER (W.).
- GEISLER (R.). Der Schraubenpropeller. Eine Darstellung seiner Entwicklung nach dem Inhalt der deutschen, amerikanischen und englischen Patentliteratur. Berlin, Springer, 1918. M. 12.00
- HAAS (A. E.). Die Grundgleichungen der Mechanik, dargestellt auf Grund der geschichtlichen Entwicklung. Leipzig, Veit, 1914. Gr. 8vo. 6 + 216 pp.
- JACOBSTHAL (W.). Mondphasen, Osterrechnung und ewiger Kalender. Berlin, Springer, 1917. M. 2.00

- JORDAN (W.). Handbuch der Vermessungskunde. Fortgesetzt von C. Reinherz. 3ter Band: Landesvermessung und Grundaufgaben der Erdmessung. 6te erweiterte Auflage. Bearbeitet von O. Eggert. Stuttgart, Metzler, 1916. M. 22.00
- . Hilfstafeln für Tachymetrie. 6te Auflage. Stuttgart, Metzler, 1917. Geb. M. 9.00
- LUDWIG (W.). Lehrbuch der darstellenden Geometrie. 1ter Teil: Das rechtwinklige Zweitafelsystem. Berlin, Springer, 1919. 8vo. 135 pp. M. 8.00
- MAKOWER (W.) et GEIGER (H.). Mesures pratiques en radioactivité. Traduit de l'anglais par E. Philippi. Paris, Gauthier-Villars, 1919. 8vo. 8 + 182 pp. Fr. 12.00
- MARBE (K.). Die Gleichförmigkeit in der Welt. München, Beck, 1916. 7 + 422 pp. M. 13.50
- . Mathematische Bemerkungen zu meinem Buch "Die Gleichförmigkeit in der Welt." München, Beck, 1916. M. 1.00
- PHILIPPI (E.). See MAKOWER (W.).
- PINEAU (J. G.). Théorie générale de l'électromagnétisme. Introduction à l'étude technique des machines électriques et de télégraphie par ondes. Paris, Dunod et Pinat, 1919. 8vo. 12 + 72 pp. Fr. 5.25
- PLANCK (M.). Thermodynamik. 5te Auflage. Leipzig, Veit, 1917. M. 10.00
- POLAK (M. W.). Bezwaren tegen de opvattingen der relativisten. Deventer, Kluwer, 1918.
- PÖSCHL (T.). Einführung in die Mechanik mit einfachen Beispielen aus der Flugtechnik. Berlin, Springer, 1917. M. 5.60
- REINHERTZ (C.). See JORDAN (W.).
- ROUGIER (L.). La matérialisation de l'énergie. Essai sur la théorie de la relativité et la théorie des quanta. Paris, Gauthier-Villars, 1919. 18mo. 12 + 148 pp. Fr. 5.25
- SCHIEFFERS (G.). Lehrbuch der darstellenden Geometrie. 1ter Teil. Berlin, Springer, 1919. 8vo. 423 pp. Geb. M. 30.60
- SCHOUTEN (J. A.). Vraagstukken vector-analysis. Delft, Waltman, 1918. 8vo. 36 pp.
- SCHWARZSCHILD (K.). Ueber das System der Fixsterne. 2te Auflage. (Naturwissenschaftliche Vorträge und Schriften, Heft 1.) Leipzig, Teubner, 1916. 43 pp. M. 1.20
- SPIELREIN (J.). Lehrbuch der Vektorrechnung nach den Bedürfnissen in der technischen Mechanik und Elektrizitätslehre. Stuttgart, K. Wittwer, 1916. 14 + 386 pp. M. 16.00
- UYEN (M. J. VAN). Het relativiteitsbeginsel. Wageningen, Veenman, 1918.
- VALAT (A.). Tableaux des moments d'inertie et des poids des poutres métalliques, calculés par le Bureau des constructions métalliques de la Compagnie des chemins de fer de l'Est. 3e édition. Paris, Gauthier-Villars, 1918. 8vo. 8 + 88 pp.

Publications of the American Mathematical Society

Transactions of the American Mathematical Society.
Published quarterly. Subscription price for annual volume, \$5.00; to members of the Society, \$3.75.

Mathematical Papers of the Chicago Congress, 1893.
Price, \$3.00; to members, \$1.50.

Evanston Colloquium Lectures, 1893. By FELIX KLEIN.
Price, 75 cents.

Boston Colloquium Lectures, 1903. By H. S. WHITE, F. S. WOODS and E. B. VAN VLECK. Price, \$2.00; to members, \$1.50.

Princeton Colloquium Lectures, 1909. By G. A. BLISS and EDWARD KASNER. Price, \$1.50; to members, \$1.00.

Madison Colloquium Lectures, 1913. By L. E. DICKSON and W. F. OSGOOD. Price, \$2.00; to members, \$1.50.

Cambridge Colloquium Lectures, 1916. Part I. By G. C. EVANS. Price, \$1.50; to members, \$1.00.

**Address all orders to AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.**

Whole No. 286

CONTENTS

	PAGE
The Twenty-Sixth Annual Meeting of the American Mathematical Society. By Professor F. N. COLE	241
The St. Louis Meeting of the American Mathematical Society. By Professor ARNOLD DRESDEN	260
Poncelet Polygons in Higher Space. By Professor A. A. BENNETT	274
On the Rectifiability of a Twisted Cubic. By Dr. MARY F. CURTIS	275
Note on Linear Differential Equations of the Fourth Order Whose Solutions Satisfy a Homogeneous Quadratic Identity. By Dr. C. N. REYNOLDS, JR.	277
An Acknowledgment of Priority. By Professor A. A. BENNETT	280
Dickson's History of the Theory of Numbers. By Professor D. N. LEHMER	281
Notes	281
New Publications	285

Changes of address of members, exchanges, and subscribers should be communicated at once to the Secretary of the American Mathematical Society, 501 West 116th Street, New York.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to The American Mathematical Society, 41 North Queen St., Lancaster, Pa., or 501 West 116th Street, New York.

The initiation fees and annual dues of members of the American Mathematical Society are payable to the Treasurer of the American Mathematical Society, Professor J. H. TANNER, Cornell University, Ithaca, N. Y.

Articles for insertion in the BULLETIN should be addressed to the
BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.

The following dates have been fixed for the meetings of the Society :

Chicago: April 9-10, 1920

New York: April 24, 1920

MAY 11 1920

BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY

**A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE**

EDITED BY

F. N. COLE	VIRGIL SNYDER	J. W. YOUNG
ALEXANDER ZIWET	D. E. SMITH	
T. LEVI-CIVITA	R. C. ARCHIBALD	

VOLUME XXVI., NUMBER 7

APRIL, 1920

**PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK**

1920

Entered at the Post Office at Lancaster, Pa., as second-class matter
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER

Whole No. 287

\$5.00 a year

GALE AND WATKEYS'S Elementary Functions and Applications

By ARTHUR S. GALE and CHARLES W. WATKEYS,
University of Rochester. (*Ready in June.*)

This freshman textbock includes plane trigonometry and parts of algebra, analytic geometry, and calculus, coherence being obtained by an analysis of relations between functions, tables of values, and graphs. Emphasis is placed upon the analysis of problems by means of graphs, and the applications include an outline of the theory underlying statistical methods. The book is intended primarily for freshmen who do not intend to specialize in mathematics, but it is also suitable as an introduction to more advanced mathematical courses. The text has been revised repeatedly in accordance with experience gained by using it in the class room for several years.

HENRY HOLT AND COMPANY

New York Boston San Francisco Chicago

PARAMETRIC EQUATIONS OF THE PATH OF A PROJECTILE WHEN THE AIR RESISTANCE VARIES AS THE n TH POWER OF THE VELOCITY.

BY PROFESSOR F. H. SAFFORD.

(Read before the American Mathematical Society April 27, 1918.)

THE differential equations to be solved are

$$(1) \quad \frac{W}{g} \frac{d^2x}{ds^2} = -Kv_c^n \frac{dx}{ds}, \quad \frac{W}{g} \frac{d^2y}{ds^2} = W - Kv_c^n \frac{dy}{ds},$$

in which v_c is the velocity along the path, W is the weight of the projectile and K and n are experimental constants, the dimensions of K being $W \cdot t^n \cdot t^n$. Obviously the X axis is taken horizontal, the Y axis vertically downward.

M. de Sparre* gives a solution for $n = 2$, making however certain approximations in the early stages, and presents his results in two cases corresponding to the paths before and after the time when the slope is unity. Greenhill† treats with much detail the case of $n = 3$.

For the general case of unrestricted n , equations (1) may be written

$$(2) \quad \begin{aligned} \frac{W}{g} \frac{d^2x}{dt^2} &= -K \left[\left[\frac{dx}{dt} \right]^2 + \left[\frac{dy}{dt} \right]^2 \right]^{(n-1)/2} \cdot \frac{dx}{dt}, \\ \frac{W}{g} \frac{d^2y}{dt^2} &= W - K \left[\left[\frac{dx}{dt} \right]^2 + \left[\frac{dy}{dt} \right]^2 \right]^{(n-1)/2} \cdot \frac{dy}{dt}, \end{aligned}$$

and next transformed by writing

$$(3) \quad \frac{dx}{dt} = v = r \cos \theta, \quad \frac{dy}{dt} = u = r \sin \theta,$$

so that r and θ are the polar coordinates of the hodograph. If the origin is taken at the point of release of the projectile and α is the angle of depression, V being the initial velocity,

* *Comptes Rendus*, volume 160, p. 584.

† *Elliptic Functions*, pp. 244-53.

then for x , y and t all zero, r and θ become V and α respectively, also

$$(4) \quad \frac{dx}{dt} = V \cos \alpha, \quad \frac{dy}{dt} = V \sin \alpha.$$

Writing $W/(gK) = S$, $W/K = T$, equations (2) become successively

$$(5) \quad S \frac{dv}{dt} = -(u^2 + v^2)^{(n-1)/2} \cdot v, \quad S \frac{du}{dt} = T - (u^2 + v^2)^{(n-1)/2} \cdot u,$$

$$(6) \quad S \frac{dr}{dt} = T \sin \theta - r^n, \quad S r \frac{d\theta}{dt} = T \cos \theta,$$

$$(7) \quad \frac{dr}{T \sin \theta - r^n} = \frac{rd\theta}{T \cos \theta} = \frac{dt}{S}.$$

From (7)

$$\frac{dr}{d\theta} - r \tan \theta = - \frac{r^{n+1}}{T \cos \theta}$$

which is reducible to the linear form, giving

$$(8) \quad r = \left[\frac{nH}{T} + \frac{n}{T} \int_0^\theta \sec^{n+1} \theta d\theta \right]^{-(1/n)} \sec \theta.$$

H in (8) is a constant of integration to be determined by the use of the initial conditions, i.e., r becomes V when θ is α , hence

$$(9) \quad V = \left[\frac{nH}{T} + \frac{n}{T} \int_0^\alpha \sec^{n+1} \theta d\theta \right]^{-(1/n)} \cdot \sec \alpha$$

or

$$(10) \quad H = \frac{W}{nKV^n \cos^n \alpha} - \int_0^\alpha \sec^{n+1} \theta d\theta.$$

(It will be noticed that H is of zero dimension.) Later expressions will be simplified by the introduction of a new constant N , which becomes unity when α is zero, or the initial direction of the motion is horizontal, viz.,

$$N = \left[1 + \frac{1}{H} \int_0^\alpha \sec^{n+1} \theta d\theta \right]^{1/n} \cos \alpha,$$

but N need not be computed since

$$(11) \quad VN = (W/(nKH))^{1/n}.$$

The reduced form of (8) is

$$(12) \quad r = VN \left[1 + \frac{1}{H} \int_0^\theta \sec^{n+1} \theta d\theta \right]^{-(1/n)} \sec \theta.$$

Again from (7)

$$(13) \quad dt = \frac{Srd\theta}{T \cos \theta} = \frac{VN}{g} \left[1 + \frac{1}{H} \int_0^\theta \sec^{n+1} \theta d\theta \right]^{-(1/n)} \sec^2 \theta d\theta.$$

Combining (3) and (13),

$$(14) \quad \begin{aligned} dx &= \frac{V^2 N^2}{g} \left[1 + \frac{1}{H} \int_0^\theta \sec^{n+1} \theta d\theta \right]^{-(2/n)} \sec^2 \theta d\theta, \\ dy &= \frac{V^2 N^2}{g} \left[1 + \frac{1}{H} \int_0^\theta \sec^{n+1} \theta d\theta \right]^{-(2/n)} \sec^2 \theta \tan \theta d\theta. \end{aligned}$$

In (14) let $\tan \theta = z$, thus introducing z as the parameter which will appear in the final equations, whence

$$(15) \quad \begin{aligned} dx &= \frac{V^2 N^2}{g} \left[1 + \frac{1}{H} \int_0^z (1 + z^2)^{(n-1)/2} dz \right]^{-(2/n)} dz, \\ dy &= \frac{V^2 N^2}{g} \left[1 + \frac{1}{H} \int_0^z (1 + z^2)^{(n-1)/2} dz \right]^{-(2/n)} z dz. \end{aligned}$$

Integration of (15) completes the formal task of obtaining the parametric equations of the trajectory, equations (10) and (11) giving the values of H and VN respectively. The practical difficulty lies in obtaining expansions of the bracketed expression in (15).

As a preliminary it is necessary to divide the computation (and the path) into two parts corresponding to values of z less than and greater than a certain value z_0 , defined by

$$(16) \quad \frac{1}{H} \int_0^{z_0} (1 + z^2)^{(n-1)/2} dz = 1,$$

a procedure which was suggested by a discussion in Chrystal's *Algebra*, volume 2, page 213. The computation of z_0 is by approximation, and if z_0 is less than 1 the binomial theorem is available. When z_0 is greater than 1 the integration in (16) must be taken in two intervals and that from 1 to z_0 effected by replacing z by s^{-1} before expanding, also using $s_0 = z_0^{-1}$.

Let P be the bracketed expression in (15), and then $\varphi(P) = [P(z)]^{-(2/n)}$ and $\varphi(Q) = [Q(s)]^{-(2/n)}$ can represent func-

tions in (15) in the respective intervals 0 to z_0 for z and 0 to s_0 for s . Arbogast's method of derivations (v. Williamson's Differential Calculus) is of assistance in making the necessary expansions especially as the computations for the two functions are identical up to a certain stage. Following Arbogast's notation closely, the work for $\varphi(P)$ is as follows:

$$(17) \quad \varphi(P) = \left[1 + \frac{1}{H} \int_0^z (1+z^2)^{(n-1)/2} dz \right]^{-(2/n)} \\ = A + Bz + Cz^2/2! + Dz^3/3! + \dots,$$

$$(18) \quad P(z) = a + bz + cz^2/2! + dz^3/3! + \dots \\ = P(0) + P'(0)z + P''(0)z^2/2! + P'''(0)z^3/3! + \dots,$$

in which z lies between 0 and z_0 , and $A, B, \dots a, b$ are to be computed. From (17) and (18) come the successive derivatives of $\varphi(P)$, also

$$(19) \quad a = P(0) = 1, \quad b = P'(0) = 1/H, \quad c = P''(0) = 0, \\ d = P'''(0) = (n-1)/H \dots,$$

$$(20) \quad A = \varphi(a), \quad B = \varphi'(a) \cdot b, \quad C = \varphi'(a) \cdot c + \varphi''(a) \cdot b^2, \\ D = \varphi'(a) \cdot d + \varphi''(a) \cdot 3bc + \varphi'''(a) \cdot b^3 \dots,$$

$$(21) \quad \varphi(a) = 1, \quad \varphi'(a) = -2/n, \quad \varphi''(a) = 2(2+n)/n^2, \\ \varphi'''(a) = -2(2+n)(2+2n)/n^3 \dots.$$

These give the desired coefficients of $\varphi(P)$, viz.,

$$(22) \quad A = 1, \quad B = -2/(nH), \quad C = 2(2+n)/(n^2H^2), \\ D = 2(n-1)/(nH) - 2(2+n)(2+2n)/(n^3H^3) \dots.$$

Similarly by the aid of (16) come

$$(23) \quad Q(s) = 2 + \frac{1}{H} \int_0^s \frac{(1+s^2)^{(n-1)/2}}{s^{n+1}} ds,$$

$$Q'(s) = -\frac{1}{H} \frac{(1+s^2)^{(n-1)/2}}{s^{n+1}},$$

$$(17)^* \quad \varphi(Q) = \bar{A} + \bar{B}(s-s_0) + \bar{C}(s-s_0)^2/2! \\ + \bar{D}(s-s_0)^3/3! + \dots,$$

$$(18)^* \quad Q(s) = \bar{a} + \bar{b}(s-s_0) + \bar{c}(s-s_0)^2/2! \\ + \bar{d}(s-s_0)^3/3! + \dots$$

(in which s lies between 0 and s_0). $\varphi(Q)$, $\varphi'(Q)$, $\dots$ are of the same form as $\varphi(P)$, $\varphi'(P)$, $\dots$. From (17)* and (18)* come

$$(19)^* \quad \bar{a} = Q(s_0) = 2, \quad \bar{b} = Q'(s_0) = -(1+s_0^2)^{(n-1)/2}/(Hs_0^{n+1}) \dots$$

$\bar{A}$, $\bar{B}$, $\bar{C}$, $\dots$ in terms of $\bar{a}$, $\bar{b}$, $\bar{c}$, $\dots$ follow from (20). From (19)*,

$$(21)^* \quad \varphi(\bar{a}) = 2^{-(2/n)}, \quad \varphi'(\bar{a}) = -2(2)^{-(2+n)/n}/n \dots,$$

giving finally $\bar{A}$, $\bar{B}$, $\bar{C}$, $\dots$ in terms of n , s_0 and H .

After obtaining A , B , $\dots$; $\bar{A}$, $\bar{B}$, $\dots$ it is necessary to substitute (17) and (17)*, corresponding respectively to the first and second portions of the path, in (15) and then to integrate. It should be noticed that $P(z_0)$ and $Q(s_0)$ are identical, so that the two portions have the same slope at their common point. For the first portion of the path

$$(24) \quad \begin{aligned} x &= \frac{V^2 N^2}{g} [Az + Bz^2/2! + Cz^3/3! + Dz^4/4! \dots], \\ y &= \frac{V^2 N^2}{g} [Az^2/2 + Bz^3/3 + Cz^4/4 \cdot 2! + Dz^5/5 \cdot 3! \dots], \end{aligned}$$

no constant of integration being added as the origin is on the path. For the second portion of the path, using $s = 1/z$ as the parameter,

$$(25) \quad \begin{aligned} x &= \frac{V^2 N^2}{g} \left[\bar{X} - \bar{A} \int_{s_0}^s \frac{ds}{s^2} - \bar{B} \int_{s_0}^s \frac{(s-s_0)}{s^3} ds \dots \right], \\ y &= \frac{V^2 N^2}{g} \left[\bar{Y} - \bar{A} \int_{s_0}^s \frac{ds}{s^3} - \bar{B} \int_{s_0}^s \frac{(s-s_0)}{s^4} ds \dots \right]. \end{aligned}$$

$\bar{X}$ and $\bar{Y}$ are constants of integration having the respective values of the two bracketed expressions in (24) when z has the value z_0 , i.e., when s is s_0 , so that the two portions of the path join.

It is evident that the accuracy of the observed constants W , K , V , n affects that of the computed constants, H , N , $\bar{X}$, $\bar{Y}$, and fixes a point beyond which the series expansions need not be carried, while the fact that x and y ultimately vary as V^2 shows that a given percentage of error in V gives approximately a double percentage of error in x and y .

UNIVERSITY OF PENNSYLVANIA,
April, 1918.

INFINITE SYSTEMS OF FUNCTIONS.

BY PROFESSOR W. E. MILNE.

THE study of infinite systems of functions is approached in this paper from an elementary point of view, and by easy steps there are derived results of considerable generality. It is found that every enumerable system of real functions whose squares are integrable in the sense of Lebesgue either is orthogonal, or possesses an adjoint, or is essentially linearly dependent. Corresponding to every normalized system of functions $\varphi_1, \varphi_2, \varphi_3, \dots$, is a set of constants $\lambda_1, \lambda_2, \lambda_3, \dots$, where $1 \geq \lambda_i \geq 0$ for every i , with the properties:

A necessary and sufficient condition that the system

- (a) be orthogonal is that $\lambda_i = 1$ for every i ,
- (b) possess an adjoint is that $\lambda_i > 0$ for every i ,
- (c) be essentially linearly dependent is that $\lambda_i = 0$ for some i .

§ 1.

For simplicity the discussion is limited to real functions of a single real variable.* The symbol Ω denotes the class of all such functions whose squares are integrable in the sense of Lebesgue in the interval (a, b) . Functions of Ω , as well as sums and products of such functions are integrable in (a, b) . The word "function" in this paper will always mean a function of class Ω , and all properties stated of a system of class Ω will be understood to hold throughout the interval (a, b) . It is assumed that the reader is familiar with the terms norm of a function, normalized system, orthogonal system, biorthogonal systems, complete systems, essential linear dependence (of a finite number of functions), convergence in the mean, etc.†

§ 2.

Let there be given an enumerable system $[\varphi]$ of normalized functions $\varphi_1, \varphi_2, \varphi_3, \dots$, of class Ω . First we investigate the

* The reader will see that the methods may be extended to more general systems.

† Definitions of all these terms are given by Brand "On infinite systems of linear integral equations," *Annals of Math.*, vol. 14 (1913), p. 101.

dependence of a finite set of these functions $\varphi_1, \varphi_2, \dots, \varphi_n$, denoted by $[\varphi]^{(n)}$, and look not merely for a test to distinguish between essential linear dependence and independence, but for a measure of the independence, or of the nearness to dependence. To formulate this precisely we assume that $\varphi_1, \varphi_2, \dots, \varphi_n$ are independent and ask how nearly can any given function φ_i of the set be expressed linearly in terms of the remaining functions of $[\varphi]^n$, where the nearness is measured by the integral of the square of the remainder. Then write

$$(1) \quad \xi_i^{(n)} = \varphi_i - (c_1\varphi_1 + \dots + c_n\varphi_n),$$

in which $c_i = 0$ and the remaining c 's are chosen to minimize the norm

$$(2) \quad \int_a^b [\xi_i^{(n)}]^2 dx = \lambda_i^{(n)}.$$

When the c 's are determined in the usual manner* and substituted into (1) the result may be put in the form†

$$(3) \quad \xi_i^{(n)} = \frac{1}{G_i^{(n)}} \begin{vmatrix} 1 & \int \varphi_1 \varphi_2 & \dots & \int \varphi_1 \varphi_n \\ \int \varphi_1 \varphi_2 & 1 & \dots & \int \varphi_2 \varphi_n \\ \dots & \dots & \dots & \dots \\ \varphi_1 & \varphi_2 & \dots & \varphi_n \\ \dots & \dots & \dots & \dots \\ \int \varphi_1 \varphi_n & \int \varphi_2 \varphi_n & \dots & 1 \end{vmatrix},$$

in which $G_i^{(n)}$ denotes the gramian of $[\varphi]^{(n)}$ with φ_i omitted, and in which the non-integrated terms $\varphi_1, \varphi_2, \dots, \varphi_n$ occupy the i th row of the determinant.

This set of functions $\xi_i^{(n)}$ has a number of interesting properties. For from equation (3) it is readily seen that

$$(4) \quad \int_a^b \varphi_i \xi_j^{(n)} dx = 0$$

if i is not equal to j , while

$$(5) \quad \int_a^b \varphi_i \xi_i^{(n)} dx = \frac{G^{(n)}}{G_i^{(n)}},$$

* See, e.g., Gram, "Über die Entwicklung reeller Funktionen in Reihen mittelst der Methode der kleinsten Quadrate," *Crelle*, vol. 94 (1883), pp. 41-73. Byerly, "Approximate representation," *Annals of Math.*, vol. 12 (1911), pp. 128-148. Böcher, "Introduction to the theory of Fourier's series," *Annals of Math.*, vol. 7 (1905), p. 81. Brand, loc. cit.

† As usual, $\int \varphi_i \varphi_j$ stands for $\int_a^b \varphi_i \varphi_j dx$.

in which $G^{(n)}$ denotes the gramian of the system $[\varphi]^n$. From (2), (3), and (5) we find

$$(6) \quad \lambda_i^{(n)} = \int_a^b [\xi_i^{(n)}]^2 dx = \frac{G^{(n)}}{G_i^{(n)}}.$$

From (5) and (6) follows

$$(7) \quad \int_a^b \varphi_i \xi_i^{(n)} dx = \lambda_i^{(n)}.$$

From (3), (4), and (7) we also get

$$(8) \quad \int_a^b \xi_i^{(m)} \xi_i^{(n)} dx = \lambda_i^{(n)} \quad (m < n).$$

From (6) and (8) we have finally

$$(9) \quad \int_a^b [\xi_i^{(m)} - \xi_i^{(n)}]^2 dx = \lambda_i^{(m)} - \lambda_i^{(n)} \quad (m < n).$$

§ 3.

So far it has been assumed that the functions of $[\varphi]^{(n)}$ are essentially linearly independent. If that is not the case, the right hand sides of equations (3), (5), and (6) may become indeterminate through the vanishing of both numerator and denominator. But the $\xi_i^{(n)}$ and $\lambda_i^{(n)}$ are still determinate. For if the function φ_m ($m < n$) is expressible linearly in terms of the functions of $[\varphi]^{(n)}$ with φ_m and φ_i omitted, the value of $\lambda_i^{(n)}$ is obviously unchanged if φ_m be dropped from the set. Therefore in forming equations (3) and (5) we shall omit every function φ_m which is expressible linearly in terms of the functions (excepting φ_i) which precede φ_m in the system $[\varphi]$. Then equations (3) to (9) hold without exception. It is evident that a necessary and sufficient condition for essential linear dependence of $[\varphi]^{(n)}$ is $\lambda_i^{(n)} = 0$ for some i .

§ 4.

Consider now the case of an infinite system of functions. From (9) it is apparent that

$$(10) \quad \lambda_i^{(m)} \geq \lambda_i^{(n)} \quad \text{if } m < n.$$

This relation, together with the fact that the λ 's are never negative, insures the convergence of $\lambda_i^{(n)}$ for each i as n

becomes infinite,

$$(11) \quad \lim_{n \rightarrow \infty} \lambda_i^{(n)} = \lambda_i \quad (i = 1, 2, \dots),$$

where

$$1 \geq \lambda_i \geq 0 \quad (i = 1, 2, \dots).$$

Recall now the following theorem of Fischer. A necessary and sufficient condition that a sequence of functions $\theta_1, \theta_2, \theta_3, \dots$ of class Ω converge in the mean to a function of this class is that to every positive ϵ however small there correspond a number N such that $\int_a^b [\theta_m - \theta_n]^2 dx < \epsilon$ whenever m and $n > N$.* Since $\lambda_i^{(n)}$ converges we see from (9) that the condition for mean convergence is satisfied for the sequence of functions $\xi_i^{(n)}$ when i is held fast and n becomes infinite. Therefore $\xi_i^{(n)}$ converges in the mean to a function ξ_i of class Ω . This system of functions $[\xi_i]$ has the property that

$$(12) \quad \int_a^b \varphi_i \xi_j dx = \begin{cases} \lambda_i & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For, by Schwarz's inequality,

$$\left[\int_a^b \varphi_i (\xi_j^{(n)} - \xi_j) dx \right]^2 \leq \int_a^b (\xi_j^{(n)} - \xi_j)^2 dx,$$

since φ_i is normalized. Since the right-hand side approaches zero as n becomes infinite we have

$$\int_a^b \varphi_i \xi_j dx = \lim_{n \rightarrow \infty} \int_a^b \varphi_i \xi_j^{(n)} dx,$$

from which, by (4), (7), and (11), we obtain (12).

§ 5.

Suppose that the system $[\varphi]$ is orthogonal. Then every gramian is unity, hence by (6) every $\lambda_i^{(n)}$ is unity, and therefore every λ_i is unity. Conversely, if every λ_i is unity, it follows from (10) that every $\lambda_i^{(n)}$ is unity, and in particular $\lambda_1^{(2)} = 1$. But this can be true only if

$$\begin{vmatrix} 1 & \int \varphi_1 \varphi_2 \\ \int \varphi_1 \varphi_2 & 1 \end{vmatrix} = 1,$$

* Fischer, "Sur la convergence en moyenne," *Comptes Rendus*, May 1907, p. 1022.

from which we find that

$$\int_a^b \varphi_1 \varphi_2 dx = 0.$$

Now any two functions of the system may be placed first, and therefore the system must be orthogonal. Therefore we have

THEOREM I. *A necessary and sufficient condition that a normalized system $[\varphi]$ be orthogonal is that $\lambda_i = 1$ for every i .*

Suppose next that $[\varphi]$ has an adjoint system $[\psi]$. For sake of symmetry we shall assume that $[\psi]$ is normalized, which requires a slight change in the usual definition of an adjoint system, for we simply require that the integral $\int_a^b \varphi_i \psi_i dx$ be greater than zero instead of requiring it to be unity. Now multiply equation (3) by ψ_i and integrate. The result is

$$\int_a^b \psi_i \xi_i^{(n)} dx = \int_a^b \varphi_i \psi_i dx.$$

Hence, by Schwarz's inequality,

$$\left[\int_a^b \varphi_i \psi_i dx \right]^2 \leq \left[\int_a^b \psi_i^2 dx \right] \left[\int_a^b \xi_i^{(n)2} dx \right],$$

from which it follows that

$$\left[\int_a^b \varphi_i \psi_i dx \right]^2 \leq \lambda_i^{(n)}$$

for every value of i . Therefore $\lambda_i > 0$ for every i . Conversely if $\lambda_i > 0$ for every i the system $[\psi]$ defined by the equations

$$(13) \quad \psi_i = \xi_i / \sqrt{\lambda_i} \quad (i = 1, 2, \dots),$$

is a normalized system adjoint to $[\varphi]$, as we see from (2) and (12). This proves

THEOREM II. *A necessary and sufficient condition that a normalized system $[\varphi]$ have an adjoint is that $\lambda_i > 0$ for every i .*

It is of interest to construct the functions ζ_i and the numbers μ_i which bear to the system $[\psi]$ defined in (13) the same relations respectively that ξ_i and λ_i bear to $[\varphi]$. It is found

upon carrying through the computations that

$$\zeta_i = \varphi_i \sqrt{\lambda_i}$$

and

$$\mu_i = \lambda_i,$$

as might have been expected from considerations of symmetry.

Another interesting property of the adjoint defined above is that of all normalized systems $[\psi]$ adjoint to $[\varphi]$ none gives

greater value to the integrals $\int_a^b \varphi_i \psi_i dx$ than does the system

defined by (13). For any adjoint whatever gives

$$\left[\int_a^b \varphi_i \psi_i dx \right]^2 \leq \lambda_i$$

but the adjoint defined by (13) requires the equality sign only.

Definition: A function θ is said to be *expressible linearly* in terms of a system $[\varphi]$ if there exists a sequence of linear combinations of φ 's converging in the mean to θ . It is evident that the functions ψ_i defined by (13) are all expressible linearly in terms of $[\varphi]$.

Consider a system $\varphi_0, \varphi_1, \varphi_2, \dots$, where φ_0 is any function of Ω and the system $\varphi_1, \varphi_2, \varphi_3, \dots$ is complete. Form the function ξ_0 . By (12) ξ_0 will be orthogonal to $\varphi_1, \varphi_2, \varphi_3, \dots$, but as this system is complete ξ_0 must be essentially zero. From the definition of $\xi_0^{(n)}$ in (1) it follows that φ_0 must be expressible linearly in terms of $\varphi_1, \varphi_2, \varphi_3, \dots$. This gives a theorem almost identical with one proved by Brand.*

THEOREM III. *If a system $[\varphi]$ is complete, every function of class Ω is expressible linearly in terms of $[\varphi]$.*

Definition: If any function φ_i of a system $[\varphi]$ is expressible linearly in terms of the remaining functions of the system, then $[\varphi]$ is said to be *essentially linearly dependent*.

If any λ_i is zero it is apparent that the system is essentially linearly dependent. On the other hand if the system is dependent, and φ_i is therefore expressible in terms of the remaining functions, the number λ_i must be zero. For if it were equal to $\epsilon > 0$, we could select a linear combination Σ_i , for which

$$\int_a^b [\varphi_i - \Sigma_i]^2 dx < \epsilon.$$

* Loc. cit., Theorem 12.

Then we may take n so great that $\xi_i^{(n)}$ contains every function in Σ_i , and then, since $\lambda_i^{(n)}$ is the least norm, we must have $\lambda_i^{(n)} < \epsilon$, and consequently $\lambda_i < \epsilon$. Hence

THEOREM IV. *A necessary and sufficient condition that a normalized system $[\varphi]$ be essentially linearly dependent is that $\lambda_i = 0$ for some i .*

Theorems II and IV give

THEOREM V. *A necessary and sufficient condition that a system $[\varphi]$ have an adjoint is that it be essentially linearly independent.*

THE UNIVERSITY OF OREGON,
November, 1919.

ON CERTAIN RELATED FUNCTIONAL EQUATIONS.

BY DR. W. HAROLD WILSON.

(Read before the American Mathematical Society December 27, 1917.)

§ 1. Introduction.

THIS paper treats of the relationships which exist between certain functional equations. In § 2, the equations

$$(1) \quad S(x - y) = S(x)C(y) - C(x)S(y),$$

and

$$(2) \quad C(x - y) = C(x)C(y) - k^2 S(x)S(y)$$

are considered individually and as a system. It is shown that (1) and (2) have their solutions in common if $C(x)$ is an even function and $S(x) \not\equiv 0$. As a consequence, it is shown that if $k \neq 0$, then

$$S(x) = [F(x) - F(-x)]/2k, \text{ and } C(x) = [F(x) + F(-x)]/2,$$

where $F(x + y) = F(x)F(y)$. If $k = 0$ and $S(x) \not\equiv 0$, $C(x) \equiv 1$ and

$$S(x + y) = S(x) + S(y).$$

The work at this point is very closely allied to that of

Jensen* on the system of equations

$$S(x+y) = S(x)C(y) + C(x)S(y),$$

$$C(x+y) = C(x)C(y) + c^2S(x)S(y).$$

This system with $c^2 = -1$ has been discussed† by Tannery, Osgood, and Van Vleck and H'Doubler.

The equation satisfied by $F(x)$ has been discussed‡ by Cauchy, Vallée Poussin, Van Vleck and H'Doubler, and others. The solution analytic over the complex x -plane has been shown to be $F(x) = e^{cx}$, where c is an arbitrary constant. Vallée Poussin (loc. cit.) has shown that if $F(x)$ is bounded in an interval $(0, \epsilon)$ along the real axis, the solution when x is real is e^{cx} , where c is an arbitrary constant. It follows, therefore, that if $F(x)$ is bounded in an interval $(0, \epsilon)$ along the real axis and in an interval $(0, \epsilon')$ along the axis of imaginaries, and if $x = u + v\sqrt{-1}$ where u and v are real, then

$$F(x) = F(u + v\sqrt{-1}) = F(u)F(v\sqrt{-1}) = e^{cu+dv},$$

where c and d are arbitrary constants, since $\varphi(v) = F(v\sqrt{-1})$ also satisfies the given equation. The solution continuous along any line in the complex x -plane also takes the last named form.

The equation $S(x+y) = S(x) + S(y)$ has been discussed by many writers of whom we may mention§ Cauchy, Darboux and Vallée Poussin. The solution $S(x)$ analytic over the complex x -plane is $S(x) = cx$, where c is an arbitrary constant. The solution bounded in an interval $(0, \epsilon)$ on the real axis and in an interval $(0, \epsilon')$ along the axis of imaginaries is, if $x = u + v\sqrt{-1}$ as above, $S(x) = cu + dv$, where c and d are arbitrary constants. The solution continuous along a line in the complex x -plane also takes this last form.

Andrade|| has found the continuous solution of

$$(3) \quad C(x+y) + C(x-y) = 2C(x)C(y),$$

* *Tidsskrift for Matematik*, vol. 2, ser. 4 (1878), p. 149.

† Tannery, *Fonctions d'une Variable*, 1886, p. 147. Osgood, *Lehrbuch der Funktionentheorie*, 1912, p. 582. Van Vleck and H'Doubler, *Transactions Amer. Math. Society*, vol. 17 (1916), p. 30.

‡ Cauchy, *Cours d'Analyse* (1821), Chapter 5. Vallée Poussin, *Cours d'Analyse infinitésimale* (1903), p. 30. Van Vleck and H'Doubler, loc. cit., p. 20.

§ Cauchy, loc. cit. Darboux, *Math. Annalen*, vol. 17 (1880), p. 56. Vallée Poussin, loc. cit.

|| *Bulletin de la Société Math. de France*, vol. 28 (1900), p. 58.

when x and y are real, by the use of integrals. Carmichael* has shown that equations (3) and

$$(4) \quad C(x+y)C(x-y) = C^2(x) + C^2(y) - 1, \quad C(0) = 1,$$

to which all equations of the form

$$\bar{C}(x+y)\bar{C}(x-y) = \bar{C}^2(x) + \bar{C}^2(y) - m^2, \quad m \neq 0,$$

can be reduced by the substitution $\bar{C}(x) = \bar{C}(0)C(x)$, have their uniform analytic solutions in common if $C(0) \neq 0$ in (3). In § 3 it is proved that all solutions of (3) and (4), except the trivial solution $C(x) \equiv 0$ of (3), are common. It is then proved that (3) and (4) have their solutions in common with the solutions of the system (1) and (2), where $S(x)$ is introduced in the discussion of (3) and (4) by the definition

$$S(x) = \frac{1}{2}[C(x-b) - C(x+b)]$$

and is chosen to be not identically zero unless $C(x) \equiv 1$.

Carmichael (loc. cit.) also discussed the uniform analytic solutions of

$$(5) \quad S(x+y)S(x-y) = S^2(x) - S^2(y).$$

The periodic solutions of equations (4) and (5) were discussed by Van Vleck and H'Doubler (loc. cit., page 30), with the added relation $C^2(x) + S^2(x) = 1$. In § 4 no relation is assumed between the functions of (4) and (5) but it is proved that $S(x)$ and $C(x)$ defined by the relation $C(x) = \frac{1}{2}[S(x+a) - S(x-a)]$, where $S(a) \neq 0$, satisfy equations (1) and (2).

In §§ 5, 6, 7, the equations

$$(6) \quad g(x+y) - g(x-y) = 2k^2S(x)S(y), \quad k \neq 0,$$

$$(7) \quad f(x+y) + f(x-y) = 2f(x)C(y),$$

and

$$(8) \quad S(x+y) - S(x-y) = 2C(x)S(y)$$

are discussed and their relationship with the preceding equations is exhibited.

Finally, in § 8, the general equation

$$(9) \quad \alpha F(x+y)F(x-y) = \beta\varphi(\mu x) + \gamma\psi(\eta y) + \delta$$

($\alpha, \beta, \gamma \neq 0$),

* *American Mathematical Monthly*, vol. 16 (1909), p. 180.

is discussed and its relation to the preceding equations is exhibited. This is analogous to Pexider's* generalization of the Cauchy equations.

It should be noted that the theorems of this paper are independent of the restrictions which may be imposed to obtain particular solutions of the equations.

§ 2. Equations (1) and (2).

THEOREM I. *If $S(x)$ and $C(x)$ satisfy equation (1) and $S(x) \not\equiv 0$, then the odd component of $C(x)$ is a constant multiple of $S(x)$, and $S(x)$ and the even component $E(x)$ of $C(x)$ satisfy equations (1) and (2) simultaneously, where*

$$k^2 = \frac{E^2(a) - E(0)}{S^2(a)} \quad \text{and} \quad S(a) \neq 0.$$

Interchange x and y . Then

$$S(y - x) = -S(x - y),$$

that is, $S(x)$ is an odd function and $S(0) = 0$. Discarding the trivial solution $S(x) \equiv 0$, there is some value a of x such that $S(a) \neq 0$.

Let $C(x) = E(x) + O(x)$, where $E(x)$ is even and $O(x)$ is odd. Equation (1) becomes

$$(1') \quad S(x - y) = S(x)E(y) + S(x)O(y) - E(x)S(y) - O(x)S(y).$$

Replace x by $-x$ and y by $-y$ in (1') and add the equation thus obtained to equation (1'). Then

$$S(x)O(y) - O(x)S(y) = 0.$$

If $y = a$,

$$O(x) \equiv \frac{O(a)}{S(a)} S(x).$$

Equation (1') now becomes

$$S(x - y) = S(x)E(y) - E(x)S(y),$$

whence

$$E(x)S(y) = S(x)E(y) - S(x - y).$$

* *Monatshefte für Mathematik und Physik*, vol. 14 (1903), p. 293.

Now replace x by $x - y$ and y by a .

$$\begin{aligned} E(x - y)S(a) &= S(x - y)E(a) - S(x - y - a) \\ &= [E(a)S(x) - S(x - a)]E(y) \\ &\quad - [E(a)E(x) - E(x - a)]S(y) \\ &= S(a)E(x)E(y) - [E(a)E(x) - E(x - a)]S(y). \end{aligned}$$

Interchange x and y and compare the equation so obtained with the last equation. It is obvious that

$$[E(a)E(x) - E(x - a)]S(y) = [E(a)E(y) - E(y - a)]S(x).$$

Let $y = a$. Then

$$E(a)E(x) - E(x - a) = k^2 S(a)S(x),$$

where

$$k^2 = \frac{E^2(a) - E(0)}{S^2(a)}.$$

Therefore

$$E(x - y) = E(x)E(y) - k^2 S(x)S(y).$$

THEOREM II. *If $S(x)$ and $C(x)$ satisfy equation (2) and if $C(x) \not\equiv \text{constant}$, then $k \neq 0$, $S(x) \not\equiv 0$, and $S(x)$ and $C(x)$ satisfy equations (1) and (2) simultaneously.*

Interchange x and y . Then

$$C(y - x) = C(x - y),$$

that is, $C(x)$ is even. If $S(x) \equiv 0$ or if $k = 0$, then

$$C(x + y) = C(x)C(-y) = C(x)C(y) = C(x - y)$$

and if $y = x$,

$$C(2x) \equiv C(0),$$

that is, $C(x)$ is identically constant (0 or 1). Setting aside this trivial case, $k \neq 0$, there is some value a of x such that $S(a) \neq 0$, and there is some value b of x such that $C(b) \neq 0$.

Set $S(x) = E_1(x) + O_1(x)$, where $E_1(x)$ is even and $O_1(x)$ is odd. Equation (2) becomes

$$\begin{aligned} C(x - y) &= C(x)C(y) - k^2 [E_1(x)E_1(y) \\ &\quad + E_1(x)O_1(y) + E_1(y)O_1(x) + O_1(x)O_1(y)]. \end{aligned}$$

Replace x by $-x$ and y by $-y$ and subtract the equation so

obtained from the last equation. Then

$$E_1(x)O_1(y) + E_1(y)O_1(x) = 0.$$

If $O_1(x) \equiv 0$, then, as above, $C(x) \equiv \text{constant}$. Discarding this trivial case, there is some value $\bar{a}$ of x such that $O_1(\bar{a}) \neq 0$. Let $x = y = \bar{a}$. Then

$$2E_1(\bar{a})O_1(\bar{a}) = 0$$

whence

$$E_1(\bar{a}) = 0.$$

Now let $y = \bar{a}$ and let x vary;

$$O_1(\bar{a})E_1(x) = 0$$

whence

$$E_1(x) \equiv 0$$

and $S(x)$ is odd and $S(0) = 0$. In (2) let $y = 0$ and $x = b$. Then $C(0) = 1$ and hence if $y = x$ in (2),

$$C^2(x) - k^2S^2(x) = 1.$$

Replace y by $-y$ in equation (2). Then

$$C(x+y) = C(x)C(y) + k^2S(x)S(y).$$

Now replace x by $x+a$ and y by a in equation (2);

$$\begin{aligned} k^2S(a)S(x+a) &= C(a)C(x+a) - C(x) \\ &= C^2(a)C(x) + k^2C(a)S(a)S(x) - C(x) \\ &= [C^2(a) - 1]C(x) + k^2C(a)S(a)S(x) \\ &= k^2S^2(a)C(x) + k^2C(a)S(a)S(x) \end{aligned}$$

whence $S(x+a) = S(a)C(x) + C(a)S(x)$.

Finally, replace x by $x-y$ and y by a in equation (2). It follows that

$$\begin{aligned} k^2S(a)S(x-y) &= C(a)C(x-y) - C(x-y-a) \\ &= C(a)C(x)C(y) - k^2C(a)S(x)S(y) \\ &\quad - C(x)C(y+a) + k^2S(x)S(y+a) \\ &= k^2S(x)[S(y+a) - C(a)S(y)] \\ &\quad - C(x)[C(y+a) - C(a)C(y)] \\ &= k^2S(a)S(x)C(y) - k^2S(a)C(x)S(y). \end{aligned}$$

Therefore

$$S(x - y) = S(x)C(y) - C(x)S(y).$$

THEOREM III. Suppose that $S(x)$ and $C(x)$ satisfy equations (1) and (2) simultaneously. If $k \neq 0$,

$$S(x) = \frac{F(x) - F(-x)}{2k} \quad \text{and} \quad C(x) = \frac{F(x) + F(-x)}{2},$$

where $F(x + y) = F(x)F(y)$. If $k = 0$, and $S(x) \neq 0$,

$$S(x + y) = S(x) + S(y) \quad \text{and} \quad C(x) \equiv 1.$$

By interchanging x and y , we see that $S(x)$ is odd and $C(x)$ is even. Replace y by $-y$. Then

$$S(x + y) = S(x)C(y) + C(x)S(y),$$

$$C(x + y) = C(x)C(y) + k^2 S(x)S(y).$$

The function $F(x) = C(x) + kS(x)$ satisfies the equation $F(x + y) = F(x)F(y)$. Now $F(-x) = C(x) - kS(x)$ and therefore

$$C(x) = \frac{1}{2}[F(x) + F(-x)] \quad \text{and} \quad S(x) = \frac{1}{2k}[F(x) - F(-x)]$$

if $k \neq 0$.

If $k = 0$, we have seen from equation (2) that $C(x) \equiv C(0)$. If $S(x) \neq 0$, then by equation (1) $C(x) \neq 0$. But if $y = 0$ in equation (2), $C(x) = C(x)C(0)$ whence if $C(b) \neq 0$, $C(0) = 1$ and (1) becomes $S(x - y) = S(x) - S(y)$. Replace y by $-y$. Then $S(x + y) = S(x) + S(y)$.

§ 3. Equations (3) and (4).

THEOREM I. If $C(x)$ satisfies equation (3) and is not identically zero, then it satisfies equation (4), and, conversely, if $C(x)$ satisfies equation (4), then it satisfies equation (3).

Since, by hypothesis, $C(x) \neq 0$, there is some value b of x such that $C(b) \neq 0$. If in (3) $x = b$, $y = 0$, then $C(0) = 1$. If in (3) $y = x$,

$$C(2x) = 2C^2(x) - 1.$$

Replace x by $x + y$ and y by $x - y$. Then

$$\begin{aligned} 2C(x + y)C(x - y) &= C(2x) + C(2y) \\ &= 2C^2(x) + 2C^2(y) - 2. \end{aligned}$$

Therefore $C(x)$ satisfies equation (4) if it satisfies equation (3) and does not vanish identically.

In (4) let $y = x$. Then

$$C(2x) = 2C^2(x) - 1.$$

Multiply equation (4) by 2 and apply the last equation. Then

$$2C(x+y)C(x-y) = C(2x) + C(2y).$$

Replace $2x$ by $x+y$ and $2y$ by $x-y$. It follows that

$$C(x+y) + C(x-y) = 2C(x)C(y),$$

that is, $C(x)$ satisfies (3) if it satisfies (4).

THEOREM II. *If $C(x)$ satisfies equation (3) or equation (4), then $C(x)$ and $S(x) = \frac{1}{2}[C(x-b) - C(x+b)]$, where b is so chosen, when possible, that $C(2b) \neq 1$, satisfy equations (1) and (2) simultaneously, where, if*

$$S(x) \neq 0, \quad k^2 = \frac{2}{C(2b) - 1}.$$

If $C(x) \equiv 0$ or $C(x) \equiv 1$ the theorem is obvious. Discarding these trivial solutions, it follows that b can be so chosen that $S(x) \neq 0$. For, if $S(x) \equiv 0$, $C(x+b) = C(x-b)$, that is, $C(2b) = C(0) = 1$. But since $C(x) \neq 1$, there is some value $2b$ of x such that $C(2b) \neq 1$.

If in (3), $x = 0$, $C(-y) = C(y)$. Furthermore

$$\begin{aligned} 2S(-x) &= C(-x-b) - C(-x+b) \\ &= C(x+b) - C(x-b) = -2S(x). \end{aligned}$$

Now

$$\begin{aligned} S(x+y) - S(x-y) &= \frac{1}{2}[C(x+y-b) + C(x-y+b)] \\ &\quad - \frac{1}{2}[C(x+y+b) + C(x-y-b)] \\ &= C(x)[C(y-b) - C(y+b)] \\ &= 2C(x)S(y). \end{aligned}$$

Subtract this equation from the equation obtained from it by interchanging x and y . Then

$$S(x-y) = S(x)C(y) - C(x)S(y).$$

Since $S(x) \neq 0$ and $C(x)$ is even, then by Theorem I, § 2,

$S(x)$ and $C(x)$ satisfy equations (1) and (2) simultaneously, where

$$k^2 = \frac{C^2(a) - 1}{S^2(a)}, \quad S(a) \neq 0.$$

But

$$\begin{aligned} S^2(a) &= \frac{1}{4}[C^2(a-b) + C^2(a+b) - 2C(a-b)C(a+b)] \\ &= \frac{1}{4}[C(a-b) + C(a+b)]^2 - C(a-b)C(a+b) \\ &= C^2(a)C^2(b) - C^2(a) - C^2(b) + 1 \\ &= [C^2(a) - 1][C^2(b) - 1]. \end{aligned}$$

Therefore

$$k^2 = \frac{1}{C^2(b) - 1} = \frac{2}{C(2b) - 1}.$$

The converse theorem is proved by adding $C(x+y)$ and $C(x-y)$, given by (2), from which it follows that $C(x)$ satisfies (3), whence if $C(x) \neq 0$ it satisfies (4).

§ 4. *The Equation (5): $S(x+y)S(x-y) = S^2(x) - S^2(y)$.*

Discarding the trivial solution $S(x) \equiv 0$, suppose a is a value of x for which $S(a) \neq 0$. Since $S(x)/m$ (where m is any constant different from zero) satisfies equation (5), we may suppose that $S(a) = 1$.

THEOREM. *If $S(x)$ satisfies equation (5), then $S(x)$ and $C(x) = \frac{1}{2}[S(x+a) - S(x-a)]$ satisfy equations (1) and (2) simultaneously, where $k^2 = C^2(a) - 1$.*

Interchange x and y . It follows that $S(x+y)S(y-x) = -S(x+y)S(x-y)$. Replace $x+y$ by a and $x-y$ by x . Then $S(-x) = -S(x)$. Now

$$\begin{aligned} 2C(-x) &= S(-x+a) - S(-x-a) \\ &= -S(x-a) + S(x+a) = 2C(x). \end{aligned}$$

Moreover $C(0) = 1$. From equation (5) it readily follows that

$$S(x+y) = S(x+y)S(a) = S^2\left(\frac{x+y+a}{2}\right) - S^2\left(\frac{x+y-a}{2}\right)$$

and

$$S(x-y) = S(x-y)S(a) = S^2\left(\frac{x-y+a}{2}\right) - S^2\left(\frac{x-y-a}{2}\right).$$

Therefore

$$\begin{aligned} S(x+y) + S(x-y) &= S(x)S(y+a) - S(x)S(y-a) \\ &= 2S(x)C(y). \end{aligned}$$

Interchange x and y and subtract the equation thus obtained from the last equation. Then

$$S(x-y) = S(x)C(y) - C(x)S(y).$$

The hypothesis of Theorem I, § 2, is fulfilled. Since $C(x)$ is even the functions $S(x)$ and $C(x)$ satisfy equations (1) and (2) simultaneously.

To prove the converse theorem when $S(x) \neq 0$, let us observe that if $y = x$ in (2), $C^2(x) - k^2S^2(x) = 1$, whence

$$\begin{aligned} S(x+y)S(x-y) &= S^2(x)C^2(y) - C^2(x)S^2(y) \\ &= S^2(x) - S^2(y), \end{aligned}$$

that is, $S(x)$ satisfies equation (5).

§ 5. Equation (6).

THEOREM. *If $g(x)$ and $S(x)$ satisfy equation (6), then $S(x)$ satisfies equation (5) and $g(x)$ can be so chosen that (by the addition of a constant) $g(2x) = 2k^2S^2(x) + 1$.*

If $S(x) \equiv 0$, $g(2x) \equiv g(0)$. Setting aside this trivial solution, it may be assumed that $S(a) \neq 0$. In fact, if $S(a) = m$, a new value of k may be chosen so that $S(a) = 1$. If $x = a$ and $y = -y$, $S(-y) = -S(y)$ and $S(0) = 0$. If $x = 0$, then $g(-y) = g(y)$. Since $\bar{g}(x) = g(x) + \bar{g}(0) - 1$ also satisfies (6), we may assume $g(0) = 1$. If $y = x$, then

$$g(2x) = 2k^2S^2(x) + 1.$$

Now in (6) replace x by $x+y$ and y by $x-y$. Then

$$S(x+y)S(x-y) = S^2(x) - S^2(y),$$

that is, $S(x)$ satisfies (5).

§ 6. Equation (7).

THEOREM. *Suppose the functions $f(x)$ and $C(x)$ satisfy equation (7) and $f(x) \neq 0$. If $f(x)$ is not an odd function, its*

even component is a constant multiple of $C(x)$, and $C(x)$ satisfies equation (3) and is not identically zero. If $f(x)$ is not an even function, $C(x)$ and the odd component $S(x)$ of $f(x)$ satisfy equations (1) and (2), where

$$k^2 = \frac{C^2(\bar{a}) - 1}{S^2(\bar{a})}, \quad S(\bar{a}) \neq 0.$$

If $f(x) \neq 0$, we may assume that $f(a) = 1$. Let $x = a$ and replace y by $-y$. Then $C(-y) = C(y)$. Let $y = 0$, $x = a$. Then $C(0) = 1$. Hence $C(x) \neq 0$. Let $f(x) = C_1(x) + S(x)$ where $C_1(x)$ is even and $S(x)$ is odd. Then

$$\begin{aligned} [C_1(x+y) + C_1(x-y)] + [S(x+y) + S(x-y)] \\ = 2C_1(x)C(y) + 2S(x)C(y). \end{aligned}$$

Add this equation to the equation obtained from it by replacing x by $-x$ and y by $-y$. Then

$$C_1(x+y) + C_1(x-y) = 2C_1(x)C(y).$$

Interchanging x and y , it is obvious that $C_1(x)C(y) = C_1(y)C(x)$. If $y = 0$, $C_1(x) = k_1C(x)$ where k_1 is a constant. Substituting this value of $C_1(x)$, it is at once apparent that $C(x)$ satisfies (3), if $k_1 \neq 0$. If $k_1 = 0$, $C_1(x) \equiv 0$ and $f(x) \equiv S(x)$.

But

$$S(x+y) + S(x-y) = 2S(x)C(y).$$

Interchanging x and y and subtracting the equation thus obtained from the last equation,

$$S(x-y) = S(x)C(y) - C(x)S(y).$$

Therefore, by Theorem I, § 2, $S(x)$ and $C(x)$ satisfy (1) and (2) simultaneously.

§ 7. Equation (8).

THEOREM. *If the functions $S(x)$ and $C(x)$ satisfy equation (8) and $S(x) \neq S(0)$, they satisfy equations (1) and (2) simultaneously, where*

$$k^2 = \frac{C^2(a) - 1}{S^2(a)}, \quad S(a) \neq 0.$$

If $S(x) \neq S(0)$, $C(x) \neq 0$ and there is a value b of x such that $C(b) \neq 0$. Let $x = b$ and replace y by $-y$. Then

$S(-y) = -S(y)$, and $S(0) = 0$. Let $y = a$ and replace x by $-x$, where a is chosen so that $S(a) \neq 0$. Then $C(-x) = C(x)$. If $y = a$ and $x = 0$, $C(0) = 1$. Subtracting equation (8) from the equation obtained from it by interchanging x and y , we have (1), whence by Theorem I, § 2, $S(x)$ and $C(x)$ satisfy (1) and (2) simultaneously.

§ 8. A General Equation.

The equation proposed for consideration is

$$(9) \quad \alpha F(x+y)F(x-y) = \beta\varphi(\mu x) + \gamma\psi(\eta y) + \delta, \quad \alpha, \beta, \gamma \neq 0.$$

If $x = y = 0$,

$$\alpha F^2(0) = \beta\varphi(0) + \gamma\psi(0) + \delta.$$

Now if $y = 0$

$$\beta\varphi(\mu x) = \alpha F^2(x) + \beta\varphi(0) - \alpha F^2(0)$$

and if $x = 0$

$$\gamma\psi(\eta y) = \alpha F(y)F(-y) + \gamma\psi(0) - \alpha F^2(0).$$

Therefore

$$F(x+y)F(x-y) = F^2(x) + F(y)F(-y) - F^2(0).$$

CASE I. $F(0) = 0$.

Let $y = x$. Then $F(x)F(-x) = -F^2(x)$ and $F(x)$ satisfies equation (5).

CASE II. $F(0) = k \neq 0$.

Let $F(x) = kC(x) + S(x)$ where $C(x)$ is even and $S(x)$ is odd. Now

$$\begin{aligned} & k^2C(x+y)C(x-y) + S(x+y)S(x-y) \\ & + kC(x+y)S(x-y) + kC(x-y)S(x+y) \\ & = k^2C^2(x) + 2kC(x)S(x) + S^2(x) + k^2C^2(y) - S^2(y) - k^2. \end{aligned}$$

Subtract from this equation the equation obtained by replacing x by $-x$ and y by $-y$. Then

$$C(x+y)S(x-y) + C(x-y)S(x+y) = 2C(x)S(x).$$

Let $y = x$. Then

$$S(2x) = 2C(x)S(x) = C(x+y)S(x-y) + C(x-y)S(x+y).$$

Replace $2x$ by $x - y$ and $2y$ by $x + y$.

$$S(x - y) = S(x)C(y) - C(x)S(y).$$

If y is replaced by $-y$,

$$S(x + y) = S(x)C(y) + C(x)S(y).$$

Therefore

$$\begin{aligned} S^2(x + y) - S^2(x - y) &= 4S(x)C(x)S(y)C(y) \\ &= S(2x)S(2y). \end{aligned}$$

Replace $2x$ by $x + y$ and $2y$ by $x - y$. Then

$$S(x + y)S(x - y) = S^2(x) - S^2(y).$$

Substituting the relations found, it follows that

$$C(x + y)C(x - y) = C^2(x) + C^2(y) - 1,$$

that is, the odd component of $F(x)$ satisfies (5) while the even component (except for the factor $F(0)$) satisfies (4).

STATE UNIVERSITY OF IOWA,
December, 1919.

THE EQUATION $ds^2 = dx^2 + dy^2 + dz^2$.

BY PROFESSOR E. T. BELL.

1. THIS equation,* being of geometrical importance, has attracted several writers, including Serret (1847), Darboux (1873, 1887), de Montcheuil (1905), Salkowski (1909), Eisenhart (1911), and Pell (1918). The simple parametric solution of de Montcheuil, which is the starting point of considerable work in differential geometry, was not noticed by Serret or Darboux. It is somewhat remarkable that the latter overlooked this solution, as he himself makes use (Surfaces,

* Full references to earlier writers are given by Eisenhart, *Annals of Math.* (2), vol. 13 (1911), pp. 17-35. Pell's paper will be found *ibid.* (2), vol. 20, pp. 142-148. The substance of the present note, with the exception of section 8, is from an unpublished A.M. thesis, presented to the University of Washington in 1908, dealing with the general algebraic problems on which solutions of this kind depend. I wish to emphasize that § 8 was written only after I had read Pell's paper.

volume III, § 584, page 11) of the 'élégant artifice de calcul' upon which the solution ultimately depends. Pell's recent solution follows from the same source. Here, without going into the theory of such equations of any degree which may be similarly solved, we shall merely show that the solutions of de Montcheuil and Pell are contained in another, from which any number of solutions of the same kind, viz., free from quadratures, may be found by inspection. Some of these may possibly be of use in geometry as are the known solutions.

We remark, however, that this equation is the simplest of a very wide class, to which the device given below is applicable. For the gist of that device lies in finding a rational integral algebraic function of several variables which reproduces itself in form with respect to multiplication, and the simplest such function, as remarked in 1202 by Fibonacci, is a sum of two squares. From one point of view the next simplest cases are Euler's four-square, and Degen's eight-square theorems. In connection with differential equations, Euler's theorem is of particular interest at the present time, giving, as will be shown elsewhere, a simple parametric solution of the fundamental equation in the Einstein-Grossmann theory of generalized relativity and gravitation.

And to call attention to an algebraic problem of importance, we may mention that the next and much more difficult class of differential equations solvable by algebraic methods analogous to that illustrated here, depends upon those algebraic forms which are transformable into a power of themselves by an algebraic substitution on the variables. The finding of all such forms, as was strongly emphasized by Eisenstein, is of the first importance arithmetically. Beyond a few isolated functions, such as the discriminant of a binary cubic (Eisenstein), and the Hessian of this discriminant (Cayley), little progress has been made toward a complete solution. The problem on several occasions attracted Cayley. It seems singular that arithmeticians have ignored Eisenstein's lead, especially after the beautiful uses which he made of his own result in his investigations on the binary cubic representations of integers. The principal object of this note is to attract attention once more to this problem, in showing that even its simplest solutions are also of use elsewhere. We may add that the algebraic problem seems to present great difficulties.

2. Fibonacci's result gives the identity

$$(1) \quad (\alpha\varphi - \beta\psi)^2 + (\alpha\psi + \beta\varphi)^2 = (\alpha\varphi + \beta\psi)^2 + (\alpha\psi - \beta\varphi)^2.$$

Hence a solution of

$$(2) \quad ds^2 = dx^2 + dy^2 + dz^2$$

is given by, ($i = \sqrt{-1}$),

$$(3) \quad \begin{aligned} s &= \int(\alpha\varphi - \beta\psi)dt, & x &= \int(\alpha\varphi + \beta\psi)dt, \\ y &= i\int(\alpha\psi + \beta\varphi)dt, & z &= \int(\alpha\psi - \beta\varphi)dt, \end{aligned}$$

in which $\alpha, \beta, \varphi, \psi$ now denote functions of a parameter t , arbitrary except as to obvious restrictions of analyticity in some domain.

3. For most purposes it is desirable to have (3) free from quadratures. Let f denote an arbitrary function of t , and $f', f'', \dots, f^{(n)}$ its successive t -derivatives. Then it is clear that, on repeated integrations by parts,

$$\int f^{(n)} t^{n-c} dt$$

wherein $c \geq n$ is an integer > 0 , is readily reducible to a form free from all integral signs. Thus

$$\int f''' t^2 dt = t^2 f'' - 2tf' + 2f, \quad \int f''' t dt = tf'' - f',$$

etc., and it is unnecessary to write out the general formula. Hence if g is a polynomial of degree $n - c$ in t , $\int f^{(n)} g dt$ is at once reducible to a form free from quadratures.

Applying this remark to (3), we choose for α, β polynomials in t of respective degrees a, b , and replace φ, ψ by $\varphi^{(c+m)}, \varphi^{(c+n)}$, where c denotes the greater of a, b , or if $a = b$, either; a, b, m, n are integers > 0 , and φ, ψ arbitrary functions of t , analytic in some domain. Upon performing the integrations by parts as indicated, (3) is reduced to a form free from quadratures. If in the result only derivatives of either function φ, ψ appear, an obvious change of notation will reduce the solution to one containing φ, ψ and their successive derivatives. Or this reduction may be obviated in the first place by assigning either a, b or m, n in advance, and noting by inspection the least values for the unassigned pair which will give a result of the desired kind.

4. A slightly different way of obtaining solutions free from quadratures depends upon the following obvious remark.

Write

$$\alpha = \sum_{r=1}^p A_r f^{(a_r)}, \quad \beta = \sum_{s=1}^q B_s g^{(b_s)},$$

and $\varphi = P$, $\psi = Q$, where A_r , B_s , P , Q are polynomials in t of respective degrees m_r , n_s , p , q , and f , g are arbitrary functions of t . Then each of $\alpha\varphi \pm \beta\psi$, $\alpha\psi \pm \beta\varphi$ in (3) is of the form

$$(4) \quad \sum_{r=1}^p \bar{A}_r f^{(a_r)} + \sum_{s=1}^q \bar{B}_s g^{(b_s)},$$

wherein $\bar{A}_r$, $\bar{B}_s$ are polynomials in t , some of which may reduce to constants, whose degrees are linear functions of the m_r , n_s , p , q . Obviously for either the a_r , b_s or the m_r , n_s , p , q pre-assigned, the other set of values may be assigned by inspection (in any infinity of ways) so that, as in § 3, the values of α , β , φ , ψ above given furnish an infinity of solutions of (2) free from quadratures.

5. To illustrate § 3, we choose $\alpha = 1$, $\beta = t$ in (3), at the same time replacing φ , ψ by φ'' , ψ'' . On integrating the resulting forms of (3) by parts as indicated, we find

$$\begin{aligned} s &= \varphi' - t\psi' + \psi, & x &= \varphi' + t\psi' - \psi, \\ iy &= \varphi - t\varphi' - \psi', & z &= \varphi - t\varphi' + \psi', \end{aligned}$$

which is de Montcheuil's solution. Discussions of the generality of this solution, with applications to geometry, will be found in the papers of de Montcheuil, Salkowski and Eisenhart.

6. In illustration of § 4 it is seen at a glance that

$$(5) \quad \begin{aligned} \alpha &= (\alpha_1 t + \beta_1) f''' + \gamma_1 g'', & \varphi &= p_1 t + q_1, \\ \beta &= (\alpha_2 t + \beta_2) f''' + \gamma_2 g'', & \psi &= p_2 t + q_2, \end{aligned}$$

where the α_i , β_i , γ_i , p_i , q_i ($i = 1, 2$) are independent of t , give a solution (3) free from quadratures. On performing the integrations by parts we get such a solution involving the ten arbitrary constants $\alpha_1, \dots, q_2$; and on assigning special values to these constants an infinity of solutions. One of the latter is Pell's. Instead of directly assigning the values of $\alpha_1, \dots, q_2$ which give Pell's solution, we shall briefly examine the more general solution, showing by other means that it actually contains Pell's.

7. Substituting the values (5) in (3), reducing the $\alpha\varphi \pm \beta\psi$, $\alpha\psi \pm \beta\varphi$ to the form (4), and integrating by parts, we get

$$(6) \quad s = (A_1 t^2 + B_1 t + C_1) f'' - (2A_1 t + B_1) f' \\ + 2A_1 f + (K_1 t + L_1) g' - K_1 g,$$

and precisely similar forms for x , $-iy$, z with the respective suffixes 2, 3, 4 in place of 1, where the twenty constants $A_1, \dots, L_4$ are as follows:

$$A_1 = \alpha_1 p_1 - \alpha_2 p_2, \quad B_1 = \alpha_1 q_1 - \alpha_2 q_2 + \beta_1 p_1 - \beta_2 p_2, \\ C_1 = \beta_1 q_1 - \beta_2 q_2, \\ A_4 = \alpha_1 p_2 - \alpha_2 p_1, \quad B_4 = \alpha_1 q_2 - \alpha_2 q_1 + \beta_1 p_2 - \beta_2 p_1, \\ C_4 = \beta_1 q_2 - \beta_2 q_1,$$

and the rest are obtained from these thus: the letters with suffix 2 from those with suffix 1 by changing all the signs to +, similarly for those with suffix 3 from 4; K_i, L_i from A_i, C_i respectively on replacing α by γ , β by γ respectively ($i = 1, 2, 3, 4$). From these twenty constants we find by a straightforward calculation that a necessary and sufficient condition that four functions of the form (6) shall give a solution of (2) is that the following indeterminate system of twelve equations be satisfied:

$$\Lambda_1^2 + \Lambda_3^2 = \Lambda_2^2 + \Lambda_4^2,$$

in which Λ represents A, C, K or L ;

$$\Lambda_1 Z_1 + \Lambda_3 Z_3 = \Lambda_2 Z_2 + \Lambda_4 Z_4,$$

where $(\Lambda, Z) = (A, B), (B, C), (C, L), (K, L),$ or (A, K) ;

$$B_1^2 + B_3^2 + 2(A_1 C_1 + A_3 C_3) = B_2^2 + B_4^2 + 2(A_2 C_2 + A_4 C_4),$$

$$A_1 L_1 + A_3 L_3 + B_1 K_1 + B_3 K_3 = A_2 L_2 + A_4 L_4 + B_2 K_2 + B_4 K_4,$$

$$B_1 L_1 + B_3 L_3 + C_1 K_1 + C_3 K_3 = B_2 K_2 + B_4 K_4 + C_2 K_2 + C_4 L_4.$$

The similarity of these conditions to those occurring in the Gaussian theory of the composition of binary quadratic forms is noticeable, and may be traced to the common source that the norm of an algebraic integer, here quadratic, is self-reproductive in form with respect to multiplication. By various artifices the set of twelve can be reduced to more

elegant forms; but as the interest of these is chiefly arithmetical, we pass them over.

8. It is easily seen that the indeterminate set of § 7 is satisfied by

$$\begin{aligned} A_1 &= B_1 = C_1 = 0; & K_1 &= 0, & L_1 &= 1; \\ A_2 &= -\frac{1}{2}, & B_2 &= 0, & C_2 &= \frac{1}{2}; & K_2 &= -1; & L_2 &= 0; \\ A_3 &= \frac{1}{2}, & B_3 &= 0, & C_3 &= \frac{1}{2}; & K_3 &= 1, & L_3 &= 0; \\ A_4 &= 0, & B_4 &= 1, & C_4 &= 0; & K_4 &= 0, & L_4 &= 1. \end{aligned}$$

Putting these in the set (6), we get a particular solution of (2),

$$\begin{aligned} s &= g', & z &= tf'' - f' + g', \\ x &= \frac{1 - t^2}{2} f'' + tf' - f - tg' + g, \\ -iy &= \frac{1 + t^2}{2} f'' - tf' + f + tg' - g, \end{aligned}$$

which is Pell's solution.

9. The extension of solutions of this kind to those containing any number of constants connected by sets of identities is immediate and need not be followed farther here. But we may briefly consider why it is possible, from the present point of view, to find solutions of (2) free from quadratures. A little consideration will show the ultimate source to lie in the fact that in (3) the integrands are bilinear in (α, β) , (φ, ψ) . This in turn is referable to the fact that Fibonacci's identity is the simplest case of Lagrange's theorem, viz., the norm of any algebraic integer reproduces itself with respect to multiplication. In Fibonacci's identity the integer is quadratic; taking the simplest case in a cubic field, we find, in exactly the same way, solutions free from quadratures for

$$dx^3 + dy^3 + dz^3 - 3dxdydz = dX^3 + dY^3 + dZ^3 - 3dXdYdZ,$$

the differential form on the left being the norm of $dx + \omega dy + \omega^2 dz$, where ω is a complex cube root of unity. Until such equations present themselves in geometry or elsewhere there is little use in writing out their solutions. But we may glance at a trigonometric device which frequently is applicable when the integrands are quadratic. It is of interest in the

simplest case because it leads to a famous solution due to Euler, also to the solution of a special case (constant coefficients), of an important equation considered by Weingarten and others. It is interesting to note that these solutions correspond to Lagrange's theorem for the general quadratic integer, so that in a sense they are generalizations of (2), which depends upon a special quadratic integer, viz., the complex unit i .

10. For a, b, k constants, φ, ψ arbitrary functions of t , we have the identities

$$\int (\varphi''' + k^2\varphi') \sin ktdt = \varphi'' \sin kt - k\varphi' \cos kt,$$

$$\int (\varphi''' + k^2\varphi') \cos ktdt = \varphi'' \cos kt + k\varphi' \sin kt,$$

$$(\varphi^2 + a\varphi\psi + b\psi^2)^2 = \Phi^2 + a\Phi\Psi + b\Psi^2,$$

where $\Phi = \varphi^2 - b\psi^2$, $\Psi = 2\varphi\psi + a\psi^2$. Hence, on putting $\alpha \equiv f''' + 4f'$, where f is an arbitrary function of t , and

$$\varphi = \sqrt{\alpha} \sin t, \quad \psi = \sqrt{\alpha} \cos t,$$

we find, on integrating by means of the first identities, that the solution of

$$ds^2 = du^2 + adudv + bdv^2$$

given by

$$s = \int (\varphi^2 + a\varphi\psi + b\psi^2)dt,$$

$$u = \int (\varphi^2 - b\psi^2)dt, \quad v = \int (2\varphi\psi + a\psi^2)dt,$$

is

$$2s = 4(b+1)f + [2(b-1)\sin 2t - 2a\cos 2t]f' \\ + [(b+1) + a\sin 2t + (b-1)\cos 2t]f'',$$

$$2u = 4(1-b)f - 2(1+b)f'\sin 2t \\ + [(1-b) - (1+b)\cos 2t]f'',$$

$$2v = 4af + (2a\sin 2t - 4\cos 2t)f' \\ + (a + 2\sin 2t + a\cos 2t)f''.$$

On putting $a = 0$, $b = 1$ we get for a solution of Euler's equation

$$ds^2 = du^2 + dv^2$$

a form equivalent to his.

UNIVERSITY OF WASHINGTON.

A PROPERTY OF PERMUTATION GROUPS ANALOGOUS TO MULTIPLE TRANSITIVITY.

BY PROFESSOR WALTER B. CARVER AND MRS. ESTELLA FISHER KING.

(Read before the American Mathematical Society December 31, 1919.)

THE binary forms

$$a_0 z_1^n + a_1 z_1^{n-1} z_2 + a_2 z_1^{n-2} z_2^2 + \cdots + a_n z_2^n$$

and

$$b_0 z_1^n + b_1 z_1^{n-1} z_2 + b_2 z_1^{n-2} z_2^2 + \cdots + b_n z_2^n$$

are said to be apolar* if

$$(1) \quad a_0 b_n - \binom{n}{1} a_1 b_{n-1} + \binom{n}{2} a_2 b_{n-2} + \cdots + (-1)^n a_n b_0 = 0.$$

Let us assume that $a_0 \neq 0$ and $b_0 \neq 0$; and let the sets of roots of the two corresponding non-homogeneous† equations

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

and

$$b_0 z^n + b_1 z^{n-1} + \cdots + b_n = 0$$

be respectively $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$. Then the condition for apolarity may be expressed in terms of these roots as follows:

$$(2) \quad S_n - \binom{n}{1}^{-1} S_{n-1} \Sigma_1 + \binom{n}{2}^{-1} S_{n-2} \Sigma_2 + \cdots \\ + (-1)^{n-1} \binom{n}{1}^{-1} S_1 \Sigma_{n-1} + (-1)^n \Sigma_n = 0,$$

where the S 's and Σ 's are the usual symmetric functions of the y 's and x 's respectively.

If now we write the product of the n differences

$$(x_1 - y_1)(x_2 - y_2) \cdots (x_n - y_n),$$

permute the x 's in all possible ways (keeping the y 's fixed), and sum the $n!$ products thus obtained, we get a result which is the same as the left-hand member of (2) except for the

* Cf. Grace and Young, *Algebra of Invariants*, p. 213; or Dickson, *Algebraic Invariants*, p. 78.

† This change to non-homogeneous form is only for convenience of expression, and is not essential to the argument.

factor $(-1)^n n!$ Thus, for $n = 3$, the condition for apolarity may be written

$$\begin{aligned} & (x_1 - y_1)(x_2 - y_2)(x_3 - y_3) + (x_2 - y_1)(x_3 - y_2)(x_1 - y_3) \\ & + (x_3 - y_1)(x_1 - y_2)(x_2 - y_3) + (x_1 - y_1)(x_3 - y_2)(x_2 - y_3) \\ & + (x_3 - y_1)(x_2 - y_2)(x_1 - y_3) + (x_2 - y_1)(x_1 - y_2)(x_3 - y_3) \\ & \equiv 6x_1x_2x_3 - 2(x_2x_3 + x_3x_1 + x_1x_2)(y_1 + y_2 + y_3) \\ & + 2(x_1 + x_2 + x_3)(y_2y_3 + y_3y_1 + y_1y_2) - 6y_1y_2y_3 = 0. \end{aligned}$$

But if instead of using all of the six permutations of the x 's, we use only the first three, we obtain exactly the same result except for the numerical factor 2; i.e., we may obtain the apolarity condition by using only the alternating group of permutations of the x 's instead of the entire symmetric group. The same is true in the case $n = 4$. But for $n = 5$ we find that the result may be obtained by using only 10 permutations* which do not form a group at all.

Two questions immediately present themselves; namely, (1) what is the smallest set of permutations which will give the apolarity condition for a given integer n ; and (2) what is the smallest *group* of permutations which may be used for the purpose. Our present interest is in a property of permutation groups which is suggested when one attempts to answer the second question.†

Definition.—A permutation group will be said to be doubly pseudo-transitive if, for some one pair of marks $x_m x_n$, there is, for any arbitrarily chosen pair of marks $x_i x_j$, at least one permutation of the group sending the pair $x_m x_n$ into the pair $x_i x_j$ in some order.

In ordinary double transitivity there would have to be one permutation sending x_m into x_i and x_n into x_j , and another sending x_m into x_j and x_n into x_i ; while in double pseudo-transitivity it is only necessary that there be a permutation sending the pair $x_m x_n$ in some order into the pair $x_i x_j$.

* They are the permutations I, (12345), (12345)², (12345)³, (12345)⁴, (2453), (4351), (4125), (5231), and (1342); which are 10 of the permutations of the group of order 20 on 5 marks.

† As to the first question, it may be said in passing that the smallest number of permutations that can be used must obviously be a common multiple of the binomial coefficients for n , and it is possibly the least common multiple. So far as the authors are aware, the question has never been fully answered.

Definition.—A permutation group on n marks will be said to be k -ply pseudo-transitive (where $1 \leq k \leq n-1$)* if, for some one set of k marks, there is, for any arbitrarily chosen set of k marks, at least one permutation of the group sending the one set into the other in some order.

It is evident that there is no difference between simple pseudo-transitivity ($k = 1$) and ordinary simple transitivity; and that if a group is k -ply transitive it is also k -ply pseudo-transitive, but not conversely.

THEOREM 1. *If a permutation group of degree n is k -ply pseudo-transitive, it is also $(n-k)$ -ply pseudo-transitive.*

THEOREM 2. *An intransitive group can not be pseudo-transitive.*

In an intransitive group of degree n it is always possible to separate the n marks into two sets such that no mark of either set is sent into any mark of the other set by any permutation of the group. Then the group could not be k -ply pseudo-transitive for $k \leq (n/2)$. For one could make a selection of k marks all from one set, and another selection containing at least one mark from the other set, and no permutation of the group would send the one selection into the other. But if a group can not be k -ply pseudo-transitive for $k \leq (n/2)$, it can not, by Theorem 1, be k -ply pseudo-transitive for any value of k .

It may be noted that ordinary transitivity of multiplicity k implies ordinary transitivity of multiplicity $1, 2, \dots, k-1$, and pseudo-transitivity of multiplicity $1, 2, \dots, k, n-k, n-k+1, \dots, n-1$; while pseudo-transitivity of multiplicity k implies only pseudo-transitivity of multiplicity $n-k, 1, n-1$, and ordinary simple transitivity.

COROLLARY. *The symmetric and alternating groups on n marks are k -ply pseudo-transitive for all values of k from 1 up to $n-1$.†*

This is an immediate consequence of the fact that these groups have ordinary transitivity of multiplicity n and $n-2$ respectively.

THEOREM 3. *If a permutation group of degree n is k -ply pseudo-transitive, its order m is divisible by $\binom{n}{k}$; and any arbitrary selection of k marks will be sent into any other*

* It is convenient to exclude the case $k = n$, as every group would be n -ply pseudo-transitive if this were included.

† A trivial exception is the alternating group of degree 2.

arbitrary selection of k marks by exactly $m/\binom{n}{k}$ permutations of the group.

This theorem, as also its proof, is analogous to the similar theorem for ordinary transitivity.

THEOREM 4. *An imprimitive group can not be k -ply pseudo-transitive for $k > 1$.*

The proof of this theorem is very similar to that of Theorem 2.

An examination of the primitive groups of degree not greater than 9 shows the following six cases of groups having pseudo-transitivity of a multiplicity other than that immediately implied by their ordinary transitivity:

Degree	Order	Transitivity	Pseudo-transitivity
7	21	Simple	Double
8	56	Double	3-ple
8	168	"	"
8	168	"	"
9	504	3-ple	4-ple
9	1512	"	"

To return to the condition for apolarity, it is evident, from theorem 3, that a necessary and sufficient condition that a group of permutations on the x 's should give this apolarity condition is that the group should be k -ply pseudo-transitive for all values of k from 1 to $n - 1$. Thus the alternating group will always serve. The smallest group that will answer the purpose for values of n up to 9 is as follows:

$n =$	2	3	4	5	6	7	8	9
order of group =	2	3	12	20	120	2520	20160	504

The group is smaller than the alternating group in the cases $n = 5, 6$, and 9.

ITHACA, N. Y.

COLLEGE ALGEBRAS.

College Algebra. By E. B. SKINNER. New York, The Macmillan Company, 1918. vi + 263 pp.

Advanced Algebra. By W. C. BRENKE. New York, The Century Company, 1917. vii + 196 pp.

A First Course in Higher Algebra. By HELEN A. MERRILL and CLARA E. SMITH. New York, The Macmillan Company, 1917. xiv + 247 pp.

RECENT textbooks written for the ordinary algebra course in college are apt to be scrutinized with other questions in mind beside the usual inquiries as to whether the traditional subject matter has been well presented and whether the problems have been so wisely selected and graded that they will be of actual assistance to the student in mastering the principles involved. For example: 1. Does the content or the manner of presentation show any modification due to the effects of those attacks upon mathematics as a required study which gave such a sharp challenge to the algebra of the high schools a few years ago? 2. Are the subjects which have been stressed in this text the topics which are usually selected for those "combination courses" in freshman mathematics which include algebra? 3. Has the desire of scientists to push back the elementary calculus into the earlier years of the college course had any effect upon this text?

In attempting to answer some of these questions regarding the three texts listed above, it has seemed advisable not to try to consider each separately, but rather to compare them. They will, of course, have much in common, but they may have enough differences to make such a comparison interesting. It is worth noting that the writers represent different types of institutions situated in different parts of the country. The authors of the first two books are professors in state universities (Wisconsin and Nebraska), while the authors of the third are in a New England college (Wellesley).

As the preface usually reveals the author's point of view, it may be well to quote here certain selected portions that seem to have especial bearing upon the questions in mind.

Professor Skinner says: "The shortening of the time given

to algebra in the secondary school, together with the great extension of the elective system and the consequent placing of mathematics in competition with a score of new subjects, has made some modification of the traditional college algebra absolutely necessary. The first important change has been to put the college algebra upon a more elementary basis. . . . The second change is one which the writer believes will in the end prove to be of great advantage, not only to algebra, but to mathematics in general. The college teacher has been obliged to exert every effort to make his work of interest to his students. To this end the subject has been made more concrete, applications to the affairs of everyday life have been emphasized, processes have been made more direct, and the road to the mathematics of the sophomore year has been shortened. . . . I have tried to emphasize the immediately practical side of algebra by drawing freely upon geometry, physics, the theory of investment, and other branches of pure and applied science for illustrative examples. An amount of space greater than usual has been devoted to the study of the functions which occur most frequently in practical work. . . . I have not hesitated to omit a number of topics that are ordinarily included in textbooks on college algebra." . . .

Professor Brenke says: . . . "The first chapter and portions of the next three chapters are reviews of elementary algebra and are intended to make a close connection with the work of the high school. The preparation of the students in the average freshman class varies so greatly that such reviews are almost indispensable to establish a common basis for further progress. Graphic methods are introduced early and freely used. This is done both for the immediate utility of these methods and to serve as an introduction to Analytical Geometry. Numerous simple applications are contained in the exercises and problems, which will serve to establish some connection between theory and practice. Logarithms are introduced in Chapter III, so as to make this method of computation available, early in the course, for the numerical valuation of unknowns from given data." . . .

Professors Merrill and Smith say: "This book is an outgrowth of the conviction of the authors that Higher Algebra, to be worthy of the name, must employ advanced methods, and that the method which chiefly marks advanced work in analysis is that of limits. In all but a few chapters the work

is based upon limits, the proofs being made as rigorous as seems advisable for immature students. . . . The ordinary Algebra course in college covers a semester's work—about forty-five class appointments. It has been found by actual use that in this time Chapters III, IV, V, VI, VII, XII can be covered, while Chapter X has been taught in connection with a course in Trigonometry." [The titles of these chapters are Determinants, Variables and their Limits, Differentiation of Algebraic Functions, Convergency of Series, Development of Functions in Series, Theory of Equations, and Logarithms (X).] "The chapters on Rational and Irrational Numbers [I and IX] are intended for reference rather than for detailed study, while the chapters on Permutations, Combinations and Probability [II], Partial Fractions [VIII], Complex Numbers [XI], and Integration [XIII] may be substituted for other chapters as subjects of study, or serve for reference in later work. No mathematical knowledge is presupposed beyond the usual course in Elementary Algebra, except that a knowledge of the meaning of the trigonometric tangent is assumed in § 105 [Chapter V] while in Chapter XI" . . . "The text has as its main ends to broaden the students' thought by introducing them as early as possible to some of the most beautiful and most fruitful methods in analysis, with a few of their results; to give an acquaintance with the simplest notions of the Calculus, . . .; and to open up to students some vision of the possibilities of later work along lines here treated only briefly. The students who are interested to do more than the assigned work have been kept constantly in mind."

In comparing these texts it may be wise to focus the attention upon three important subjects; logarithms, functions, and graphical representation, which would certainly be included in a "combination course" in freshman mathematics.

The space devoted to logarithms varies from twelve pages (Brenke) to twenty-six (Merrill and Smith). Professor Brenke adds a four-place table of logarithms of numbers and Professor Skinner has this and five other tables at the end of his book (powers and roots, important constants, compound interest and compound discount tables, and American experience table of mortality). The relative position of the first presentation of the subject is more significant than the number of pages. In the first book it is the fourth of eighteen chapters, being pre-

ceded by introduction, algebraic identities, and powers and roots. In the second text the third of thirteen chapters is on logarithms and the binomial theorem, the first two being on a review of the four fundamental operations and on theory of exponents and surds and imaginaries. Both these texts return to the subject of logarithms later. The position which the chapter on logarithms occupies in the third text has already been noted. In the first two texts, where the aim is to give the student an early introduction to the use of logarithms in numerical calculations, the first presentation contains the usual subject matter on common logarithms with sets of exercises including not only the purely arithmetic variety but also applied problems from geometry, physics, and compound interest. The first text adds two pages, with diagrams, on the logarithmic scale and the slide rule. In Chapter XIV (Progressions and Compound Interest) in the section on the compound interest law, the student is introduced to e . The last two sections of the last chapter (infinite series) are on the logarithmic series and series as a means of computation. In the second text natural logarithms are mentioned in a note to a problem at the end of Chapter VIII (infinite series) and in the next chapter (computations, approximations, differences and interpolation) there are sections on the computation of natural logarithms and common logarithms.

In the third text the chapter on logarithms opens with a long quotation from Napier's preface to his *Descriptio*, which will be much appreciated by students,—especially his reference to the “many slippery errors.” It might be noted in passing that there is an appropriate quotation at the head of every chapter in this book. The treatment of logarithms in this text stands out in strong contrast to the method of the other two. After giving the definitions and the simple theorems, the authors say: “These theorems show that if the logarithms of all numbers to some base can be found, arithmetical processes can be greatly shortened.” Then, after discussing the use of logarithms in solving exponential equations and adding historical notes on Napier and Briggs, they say, “The following sections give a method of computing the logarithms of numbers to any desired base.” (It must be recalled that limits, derivative of an algebraic function, and series have been studied before this chapter.) Fifteen pages later the student is told, “The results obtained by these and other methods of com-

puting logarithms are arranged in various forms, . . . modern tables are extremely easy to use after a little practice. As full instructions for using the tables are ordinarily included in each book, only a few general principles are given here." Two pages later there is a list of eleven exercises. This book does not contain a table of logarithms of numbers.

The place which graphic methods occupy in Professor Brenke's book is well stated in his preface. In Chapter IV (Linear Equations) after the algebraic solution of an equation in one unknown, the author discusses coordinates, the graph of the "expression" $2x - 1$ and the theorem "The graph of the equation $y = ax + b$ is a straight line." Throughout this chapter and the next, the author uses this plan of following the algebraic solution by the graphic. In Chapter V under "the graphic solution of two simultaneous equations in two variables, the one linear, the other quadratic," the student is given the standard equation of the circle, ellipse, parabola, and hyperbola and the equation of the rectangular hyperbola and the general equation of the second degree. He meets the terms asymptotes and conic section. It is not until the next chapter (Ratio, Proportion, Variation) that the term *function* is defined. According to the index, this is the only time that the term is used in the text. (In passing, it might be noted that the third book contains no index and that the first has a good one, and that the brief index in this text is quite lacking in one place at least, for it does not contain the word graph (or graphical), although the subject of graphs is so much stressed.)

In the first text the chapter that follows the one on logarithms is entitled "Functions of a Variable — Graphical Representation." On the first page, after a careful explanation of the term function, there is the statement, "The notion of a function is one of the most important ideas with which mathematics has to do." After giving many varied illustrations, the author explains coordinates and the graph of a function and then deals with the linear integral function (and its graph), uniform motion, rational integral quadratic function, maximum and minimum values of quadratic functions, the power function, variation, discontinuous functions, statistical graphs, interpolation of values of functions, zeros and infinities of a function and implicit functions. The greater part of the next chapter (Quadratic Equations with One Unknown) is

devoted to the algebraic solution and the relation between roots and coefficients. But in the section on "geometric interpretation," its relation to the quadratic function of the last chapter is brought out. In Chapters VII, VIII, IX, and XI (systems of linear equations, and systems containing non-linear equations, inequalities, polynomials, and equations of any degree) the notion of a function is never lost sight of. It is to be noted that graphs are employed as needed in all these chapters, while in Chapter VIII the standard forms for the different species of conics are given.

In the third text the three and a half page introduction which is inserted between the preface and the first chapter gives information regarding the authors' attitude toward the place of graphical representation in their text. The introduction opens with the statement: "The following topics from elementary algebra are inserted for reference." [1. Graphical representation, (two pages, including the graph of $y = x^2 - 4x + 3$). 2. Roots of quadratic equations. 3. Binomial theorem. 4 and 5. Formulas for arithmetic and geometric progressions.] Assuming that graphical representation is a subject with which the student is already sufficiently familiar, it is employed in the text wherever it is deemed useful. This book resembles the first in giving a central position to the notion of the function. But the methods of presentation in the two books are quite different. According to the selection of material suggested in the preface, Professors Merrill and Smith begin with determinants and then, turning to variables and their limits, introduce the function on the second page of the new chapter and proceed to a discussion of algebraic functions, limits and infinitesimals (including theorems on limits), infinity, and continuity of functions. In the next chapter (twenty-six pages) the student is led up to the idea of the derivative and learns of theorems on derivatives and studies successive differentiation, geometric interpretation of the derivative, applications, and maximum and minimum values.

It is natural at this point to inquire whether either of the other texts goes into the question of derivatives. In the index of the second book the word derivative does not occur and the word limit appears but once,—the limit of a variable is defined in the chapter on infinite series. In the first text the word limit is used for the first time in connection with infinite

geometric progressions (Chapter XIV). In Chapter XVII (Sequences and Limits), limits of variables are defined and discussed. This fourteen page chapter contains also explanations of the derivative of a power function, of the geometric interpretation of the derivative, and of maxima and minima.

A more detailed study of these texts will, of course, reveal other interesting material, and it will deepen the reader's conviction that the second book follows the traditional lines more closely and that, while the authors of the first and third texts differ radically from one another in their aims, each has succeeded in presenting an attractive book that has a distinct place.

E. B. COWLEY.

NOTES.

THE twenty-seventh summer meeting and ninth colloquium of the American Mathematical Society will be held at the University of Chicago, extending through the week September 6-11. The regular sessions of the Society and also those of the Mathematical Association of America will occupy the first days of the week, without conflict of hours. The joint dinner will be held on Tuesday evening. The Colloquium will open on Wednesday and extend to Saturday noon. It will consist of two courses of five lectures each by Professors G. D. BIRKHOFF, of Harvard University, and F. R. MOULTON, of the University of Chicago. Titles and principal topics of the lectures follow:

Professor BIRKHOFF: "Dynamical systems." The last forty years have witnessed fundamental advances in the theory of dynamical systems, achieved by Hill, Poincaré, Levi-Civita, Sundman, and others. The lectures will expound the general principles underlying these advances, and will point out their application to the problem of three bodies as well as their significance for general scientific thought. The following topics will be treated:

Physical, formal, and computational aspects of dynamical systems. Types of motions such as periodic and recurrent motions, and motions asymptotic to them. Interrelation of types of motion with particular reference to integrability and

stability. The problem of three bodies and its extension. The significance of dynamical systems for general scientific thought.

Professor MOULTON: "Certain topics in functions of infinitely many variables." I. On the definition and some general properties of functions of infinitely many variables. II. On infinite systems of linear equations. III. Infinite systems of implicit functions. IV. Infinite systems of differential equations. V. Applications to physical problems.

It is announced that an international congress of mathematicians from the Entente and approved neutral countries will be held at Strasbourg, opening on September 22, 1920. The congress will divide into the usual four sections, with convenient subdivisions. A detailed programme, with information regarding transportation, hotels, receptions and excursions will be issued later. The membership fee is sixty francs, payable to M. Valiron, treasurer of the congress, 52 Allée de la Robertsau, Strasbourg. Each member will receive a copy of the published proceedings of the congress, which will contain at least a summary of papers and transactions. A member can obtain additional cards for his family at thirty francs each.

Information regarding the congress can be obtained from Professor G. KOENIGS, general secretary of the French national committee of mathematicians, 96 Boulevard Raspail, Paris. Titles of papers intended for presentation at the congress should be in Professor Koenigs' hands by July 1.

THE opening (January) number of volume 42 of the *American Journal of Mathematics* appears in a new dress. Sylvester's quarto page, luxurious margins, and large type are now a memory of the past. The number contains the following papers: "Groups of order 2^m " which contain a relatively large number of operators of order two," by G. A. MILLER; "The Green's function for a plane contour," by H. D. FRARY; "On the solution of certain types of linear differential equations in infinitely many variables," by W. G. SIMON; "Periodic orbits on a surface of revolution," by DANIEL BUCHANAN.

AT the meeting of the London mathematical society on

January 15, the following papers were read: By P. A. MACMAHON, "The divisors of numbers"; by H. STEINHAUS, "Fourier coefficients of bounded functions"; by S. P. OWEN, "The lag of a thermometer in a medium whose temperature is a linear function of the time."

At the meeting of February 13, papers were read by G. S. LEBEAU, "A property of polynomials whose roots are real"; by the late E. K. WAKEFORD, "Canonical forms"; by E. LANDAU and A. OSTROWSKI, "A problem of Diophantine analysis"; by G. H. HARDY and J. E. LITTLEWOOD, "The zeros of Riemann's zeta function."

At the meeting of the Edinburgh mathematical society on February 13, the following papers were read: By T. M. MACROBERT, "On the modified Bessel function $K_n(z)$ "; by E. M. HORSBURGH, "A radial transparent scale for use with an abacus"; by W. A. BARCLAY, "Nomograms for financial formulae."

At the meeting of March 12, papers were read by C. TWEDDIE, "The life of James Stirling, the Jacobite mathematician, with exhibition of autograph letters by N. Bernoulli, Clairaut, Cramer, Euler, Maclaurin, and Stirling"; by J. MITCHELL, "On the arrangement of the signs of the terms in a certain double series given by Arndt."

THE following university courses in mathematics are announced for the summer session:

UNIVERSITY OF CHICAGO (first term, June 1 to July 28; second term, July 29 to September 3). By Professor E. H. MOORE: Hermitian matrices of positive type in general analysis, four hours; Limits and series, four hours, first term only.—By Professor G. A. BLISS: Functions of a complex variable, four hours.—By Professor L. E. DICKSON: Theory of matrices and bilinear quadratic forms, four hours; Theory of numbers, four hours.—By Professor M. W. HASKELL: Projective geometry, four hours; Topics in analytic geometry, five hours.—By Professor J. W. A. YOUNG: Theory of equations, four hours.—By Professor E. W. CHITTENDEN: Differential equations, four hours.—by Professor W. D. MACMILLAN: Celestial mechanics II, four hours.

THE Society of sciences of Göttingen announces the following prize problem for the year 1921: Riemann conjectured

that all the non-negative zeros of the zeta function have their real part equal to one half; a proof or disproof of this is desired, or other important discoveries with regard to the location of the roots of the Riemann zeta function or related functions. Competing memoirs should be sent to the Society before August 1, 1921. The prize is 1000 marks.

THE Prince Jablonowski Society of Leipzig announces as the subject of its prize memoir for 1921 a certain type of generalization of the addition theorem for elliptic functions. Further information can be obtained from the Society, to which competing memoirs should be sent before October 31, 1921. The prize is 1500 marks.

THE prize of the Peter Wilhelm Müller Foundation, which is awarded every three years for achievement in some one of the arts or sciences, was assigned to mathematics in 1918, and was divided between D. HILBERT, for his entire mathematical work, and A. EINSTEIN, for his work in mechanics, especially in relativity and gravitation theory.

PROFESSORS O. PERRON, of the University of Heidelberg, and I. SCHUR, of the University of Berlin, and Dr. H. WIELEITNER have been elected members of the K. Leopoldinisch-Carolinischen Deutschen Akademie der Naturforscher in Halle.

PROFESSORS L. POCHHAMMER, of the University of Kiel, and A. WANGERIN, of the University of Halle, have retired from active teaching.

A JOINT committee has been appointed by Cambridge University and the Royal Society of London for the purpose of taking steps to secure an appropriate memorial to the late Lord RAYLEIGH.

MR. W. J. HARRISON has been appointed university lecturer in mathematics at Cambridge University.

PROFESSOR JACQUES HADAMARD delivered a series of three lectures on "The first years of the work of Poincaré" at the Rice Institute in March.

THE French government has conferred the decoration "Officier de l'instruction publique" upon Professor E. B. VAN VLECK, of the University of Wisconsin, in recognition of his services as teacher and investigator and for his work during the war.

PROFESSOR ALFRED BAKER, of the department of mathematics of the University of Toronto, has retired, after forty-four years of service. Professor A. T. DELURY succeeds him as head of the department. Mr. I. R. POUNDER has been promoted to an assistant professorship of mathematics.

PROFESSOR W. W. LANDIS, of Dickinson College, has returned to his academic work with the honorary rank of major in the Italian army, conferred on him in recognition of his work as a Y. M. C. A. secretary in Italy during the war.

AT Cornell University, Professor JAMES McMAHON has been granted leave of absence for the year 1920-1921, Professor ARTHUR RANUM for the first term, and Professor F. R. SHARPE for the second term. Mr. D. S. MORSE, of Union College, Professor W. L. G. WILLIAMS, of William and Mary College, and Mr. H. PORITZKY have been appointed instructors in mathematics.

PROFESSOR J. L. JONES, of Syracuse University, has been appointed head of the department of mathematics at Akron University.

AT the College of the City of New York, Mr. J. A. BREWSTER has been promoted to an assistant professorship of mathematics, and Mr. W. A. WHYTE to an instructorship in mathematics.

AT the University of Maine, Mr. A. S. PRATT, of Brown University, has been appointed instructor in mathematics, and Mr. J. P. BALLANTINE, of Harvard University, instructor in mathematics and physics.

MR. Y. H. HO, fellow in mathematics at Cornell University, died February 22, 1920, at the age of twenty-seven years. He was elected to membership in the American Mathematical Society in 1919.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- ALBANESE (G.). Sopra alcune questioni di geometria algebrica. Pisa, tip. succ. fratelli Nistri, 1919. 8vo. 29 pp.
- BACHMANN (P.). Das Fermatproblem in seiner bisherigen Entwicklung. Berlin und Leipzig, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1919. 8 + 160 pp. M. 12.00
- BRESLICH (E. R.). Correlated mathematics for junior colleges. Chicago, University of Chicago Press, 1919. 13 + 301 pp. \$1.25
- BROCARD (H.). et LEMOYNE (T.). Courbes géométriques remarquables planes et gauches (courbes spéciales). Volume 1. Paris, Vuibert, 1919. 8vo. 8 + 451 pp. Fr. 18.00
- BROWNE (R. T.). The mystery of space. A study of the hyperspace movement in the light of evolution of new psychic faculties and an enquiry into the genesis and essential nature of space. New York, Dutton, 1919. 8vo. 17 + 395 pp. \$4.00
- BURALI-FORTI (C.). Logica matematica. 2a edizione, interamente rifatta. (Manuali Hoepli.) Milano, Hoepli, 1919. 16mo. 32 + 483 pp. L. 9.50
- CARVALHO (A. S. C. G. DE). A teoria das tangentes antes da invenção do cálculo diferencial. (Diss.) Coimbra, Imprensa da Universidade, 1919. 99 pp.
- COHEN-KYSER (A.). Rückläufige Differenzierung und Entwicklung. Leipzig, Barth, 1919. M. 3.00
- CRANTZ (P.). Analytische Geometrie der Ebene zum Selbstunterricht. 2te Auflage. Leipzig, Teubner, 1919.
- DAVISON (C.). The elements of analytical conics. London, Cambridge University Press, 1920. 7 + 238 pp. 10s.
- GÉRARD (A. E.). See THOMSON (S. P.).
- LEMOYNE (T.). See BROCARD (H.).
- LÖFFLER (E.). Ziffern und Ziffernsysteme. 2ter Teil: Die Zahlzeichen im Mittelalter und in der Neuzeit. Leipzig, Teubner, 1919. 16mo. 59 pp. Cartoniert.
- MANTZ (O.). Zur Basisbestimmung der Napierschen und Bürgischen Logarithmen. Basel, Kreis, 1919. 8vo. 49 pp.
- MAYO (C. H. P.). Elementary calculus with answers. London, Rivington, 1919. 20 + 345 + 39 pp. 10s.
- SILBERSTEIN (L.). Elements of vector algebra. London, Longmans, 1919. 7 + 42 pp. 5s.
- THOMSON (S. P.). Le calcul intégral et différentiel à la portée de tout le monde. Traduit de l'anglais par A. E. Gérard. Paris, Dunod et Pinat, 1919. 12mo. 8 + 290 pp. Fr. 12.00

II. ELEMENTARY MATHEMATICS.

AUERBACH (M.) and WALSH (C. B.). *Plane geometry*. (Lippincott's School Text Series.) Philadelphia, Lippincott, 1920. 8vo. 16 + 383 pp. \$1.32

BAFFI (C.). *Elementi di algebra, ad uso della prima classe normale, colla teoria della radice quadrata e cubica. 1a ristampa della 2a edizione riveduta ed ampliata.* (Biblioteca di Scienze fisiche, matematiche e naturali: Collezione Paravia.) Torino, ditta G. B. Paravia (Milano, stamp. Lombarda, di L. Mondaini), 1919. 8vo. 7 + 192 pp. L. 4.50

—. *Elementi di aritmetica razionale, ad uso della seconda classe normale. Ristampa della 2a edizione riveduta.* (Biblioteca di Scienze fisiche, matematiche e naturali: Collezione Paravia.) Torino, ditta G. B. Paravia (Milano, stamp. Lombarda, di L. Mondaini), 1919. 8vo. 8 + 187 pp. L. 5.00

BARONI (E.) e FONTEBASSO (P. A.). *Geometria per il ginnasio inferiore e scuole complementari.* Milano-Roma-Napoli, soc. ed. Dante Alighieri, di Albrighi, Segati e C. (Città di Castello, S. Lapi), 1919. 8vo. 12 + 179 pp. L. 2.50

BORTOLOTTI (E.). *Nozioni di geometria per le scuole tecniche: guida agli esami di geometria. 7a edizione riveduta.* Milano-Roma-Napoli, soc. ed. Dante Alighieri, di Albrighi, Segati e C. (Città di Castello, S. Lapi), 1919. 8vo. 282 pp. L. 4.00

CRACKNELL (A. G.). See WORKMAN (W. P.).

DIXITDOMINUS (C.). *Teoria grafica dimostrativa sulla trisezione degli angoli piani.* Palermo, tip. fratelli Vena, 1919. 8vo. 27 pp. + 4 tavole.

FONTEBASSO (P. A.). See BARONI (E.).

GRAY (J. C.). *Number by development. Parts 1 and 2.* Philadelphia, Lippincott, 1919. 12 + 486 + 20 + 514 pp.

HARANG (F.). *Éléments de trigonométrie suivis d'une instruction sur la règle à calcul.* Paris, Dunod et Pinat, 1919. 8vo. 6 + 138 pp. Fr. 9.75

WALSH (C. B.). See AUERBACH (M.).

WORKMAN (W. P.) and CRACKNELL (A. G.). *The school geometry: matriculation edition.* London, Clive, 1919. 11 + 348 pp. 4s. 6d.

III. APPLIED MATHEMATICS.

ANGLES (J. W.). *Mensuration for marine and mechanical engineers.* London, Longmans, 1919. 27 + 162 pp. 5s.

ARCHIMEDES. See CHRISTENSEN (S. A.).

BARRAUD (A.). *Tables pour le calcul et le tracé des courbes.* Paris, Béranger, 1919. 135 pp. Cartonné. Fr. 8.40

BLOCH (L.). *Précis d'électricité théorique.* Paris, Gauthier-Villars, 1919. 8vo. 6 + 476 pp. Fr. 30.00

BORCHARDT (W. G.). *School mechanics. Part 1: School statics.* London, Rivington, 1919. 6s.

BOUTHILLON (L.). *La théorie et la pratique des radiocommunications. I: Introduction à l'étude des radiocommunications.* Paris, Delagrave, 1919. 8vo. 195 pp. Fr. 20.00

BROWN (H. T.). Cinq cent sept mouvements mécaniques. Traduit de l'anglais par H. Stévant. Nouveau tirage. Paris, Gauthier-Villars, 1919. Fr. 3.60

CHRISTENSEN (S. A.). Archimedes Sandregning og Aristarchos fra Samos. "Solens og Maanens Størrelse og Afstand." Nykøbing Katredralskoles Aarsprogram, 1919.

ECCLES (J. R.). Advanced lecture notes on light. Cambridge, University Press, 1919. 141 pp.

GÜMLICH (E.). Leitfaden der magnetischen Messungen mit besonderer Berücksichtigung der in der Physikalisch-Technischen Reichsanstalt verwendeten Methoden und Apparate nebst einer Uebersicht über die magnetischen Eigenschaften ferromagnetischer Stoffe. Braunschweig, Vieweg, 1918. 8vo. 8 + 228 pp.

HARLEY DI SAN GIORGIO (O.). Formulario dell'ingegnere. (Collezione Lattes.) Torino, S. Lattes (V. Bona), 1919. 16mo. 174 pp. L. 6.50

JUDGE (A. W.). The design of aeroplanes. London, Macmillan, 1919. 8vo. 10 + 242 pp. 10s. 6d.

LINOTTE (M.). See MARTIN (J.).

LORIA (G.). See TORRICELLI (E.).

MARTIN (J.) et LINOTTE (M.). Dessin industriel. Cours pratique de projections. Paris, Dunod et Pinat, 1919. 4to. 72 pp. Fr. 14.00

ROTHE (R.). Darstellende Geometrie des Geländes. 2te verbesserte Auflage. (Mathematische-physikalische Bibliothek, Nr. 35-36.) Leipzig, Teubner, 1919. 92 pp.

RUNGE (C.). Graphische Methoden. 2te Auflage. Leipzig, Teubner, 1919. 130 pp.

RUTOT (A.). La conception nouvelle de l'univers d'après les données de la science moderne. Bruxelles, Hayez, 1919. 18mo. 96 pp.

SCHMID (T.). Darstellende Geometrie. 1ter Band. 2te Auflage. (Sammlung Schubert, Nr. 65.) Berlin und Leipzig, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1919. 278 pp. M. 15.40

STÉVART (H.). See BROWN (H. T.).

TORRICELLI (E.). Opere. Edited by G. Loria and G. Vassura. 3 volumes. Faenza, G. Montanari, 1919. Volume 1, part 1, 38 + 407 pp.; volume 1, part 2, 482 pp.; volume 2, 320 pp.; volume 3, 521 pp. L. 60.00

VASSURA (G.). See TORRICELLI (E.).

Publications of the American Mathematical Society

Transactions of the American Mathematical Society.

Published quarterly. Subscription price for annual volume, \$5.00; to members of the Society, \$3.75.

Mathematical Papers of [the Chicago Congress, 1893.

Price, \$3.00; to members, \$1.50.

Evanston Colloquium Lectures, 1893. By FELIX KLEIN.

Price, 75 cents.

Boston Colloquium Lectures, 1903. By H. S. WHITE, F.

S. WOODS and E. B. VAN VLECK. Price, \$2.00; to members, \$1.50.

Princeton Colloquium Lectures, 1909. By G. A. BLISS

and EDWARD KASNER. Price, \$1.50; to members, \$1.00.

Madison Colloquium Lectures, 1913. By L. E. DICKSON

and W. F. OSGOOD. Price, \$2.00; to members, \$1.50.

Cambridge Colloquium Lectures, 1916. Part I. By G.

C. EVANS. Price, \$1.50; to members, \$1.00.

Address all orders to AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.

Whole No. 287

CONTENTS

	PAGE
Parametric Equations of the Path of a Projectile When the Air Resistance Varies as the n th Power of the Velocity. By Professor F. H. SAFFORD - - - - -	289
Infinite Systems of Functions. By Professor W. E. MILNE -	294
On Certain Related Functional Equations. By Dr. W. H. WILSON - - - - -	300
The Equation $ds^2 = dx^2 + dy^2 + dz^2$. By Professor E. T. BELL	312
A Property of Permutation Groups Analogous to Multiple Transitivity. By Professor W. B. CARVER AND MRS. ESTELLA F. KING - - - - -	319
College Algebra. By Professor ELIZABETH B. COWLEY - -	323
Notes - - - - -	329
New Publications - - - - -	334

Changes of address of members, exchanges, and subscribers should be communicated at once to the Secretary of the American Mathematical Society, 501 West 116th Street, New York.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to The American Mathematical Society, 41 North Queen St., Lancaster, Pa., or 501 West 116th Street, New York.

The initiation fees and annual dues of members of the American Mathematical Society are payable to the Treasurer of the American Mathematical Society, Professor J. H. TANNER, Cornell University, Ithaca, N. Y.

Articles for insertion in the BULLETIN should be addressed to the
BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.

The following dates have been fixed for the meetings of the Society :

The Twenty-Seventh Summer Meeting and Ninth Colloquium of the Society will be held at the University of Chicago, extending through the week September 6-11, 1920.

IVARD COLLEGE
MAY 17 1920
LIBRARY

BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY

**A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE**

EDITED BY

F. N. COLE	VIRGIL SNYDER	J. W. YOUNG
ALEXANDER ZIWET	D. E. SMITH	
T. LEVI-CIVITA	R. C. ARCHIBALD	

VOLUME XXVI., NUMBER 8

MAY, 1920

PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK
1920

Entered at the Post Office at Lancaster, Pa., as second-class matter
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER

Whole No. 288

\$5.00 a year

Digitized by Google

In the new

[HAWKES—LUBY—TOUTON



 PLANE GEOMETRY

The authors have applied to geometry those principles of modern teaching that have made their algebras one of the most successful series of the generation. They develop sure scientific imagination.

Nearly Ready

GINN AND COMPANY

70 Fifth Avenue

NEW YORK

The Mathematical Library

of the late

Professor Adolph Hurwitz, of Zurich

is offered for sale

420 Volumes of Mathematical Works

380 Volumes of Journals

5,000 Separate Papers

Price, \$1,200, plus packing and freight charges

For Catalogue address

Mrs. Hurwitz, Bächtoldstrasse, 11, Zurich

THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE two hundred and eighth regular meeting of the Society was held at Columbia University on Saturday, February 28, extending through the usual morning and afternoon sessions. The attendance included the following twenty-eight members:

Dr. J. W. Alexander, Professor R. C. Archibald, Professor A. A. Bennett, Professor Pierre Boutroux, Professor B. H. Camp, Professor F. N. Cole, Professor L. P. Eisenhart, Professor T. S. Fiske, Professor W. B. Fite, Dr. T. H. Gronwall, Professor C. C. Grove, Professor Edward Kasner, Dr. E. A. T. Kircher, Mr. Harry Langman, President E. O. Lovett, Professor H. H. Mitchell, Mr. George Paaswell, Dr. G. A. Pfeiffer, Professor H. W. Reddick, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor L. P. Siceloff, Dr. J. M. Stetson, Professor Oswald Veblen, Mr. H. E. Webb, Professor J. H. M. Wedderburn, Professor J. K. Whittemore.

Vice-President R. G. D. Richardson occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. F. J. Burkett, Pennsylvania State College; Mr. A. D. Campbell, Cornell University; Dr. Y. R. Chao, Cornell University; Professor R. E. Gilman, Brown University; Mr. D. C. Kazarinoff, University of Michigan; Dr. Norman Miller, Queen's University; Dr. G. M. Robison, Cornell University; Professor Jung Sun, Peking Academy; Dr. W. H. Wilson, State University of Iowa; Dr. S. D. Zeldin, Massachusetts Institute of Technology. Six applications for membership in the Society were received.

Professor Oswald Veblen was appointed to fill the unexpired term of Professor E. W. Brown, resigned, as representative of the Society in the Division of Physics of the National Research Council. The other representatives are Professor L. E. Dickson, reporter, and Professor H. S. White. These three representatives, the President and the Secretary of the Society, and Professors J. L. Coolidge and L. P. Eisenhart form a committee to consider the formation of an American Section of an International Mathematical Union. Professor Dickson is chairman of this committee.

Steps were taken to submit the question of the incorporation of the Society to the members at the April meeting.

The following papers were read at the February meeting:

(1) Professor JOSEPH LIPKA: "On the general problem of dynamics."

(2) Dr. A. R. SCHWEITZER: "On the iterative properties of the algebra of logic."

(3) Dr. A. R. SCHWEITZER: "On improper pseudogroups, with application to the abstract field."

(4) Mr. G. H. HARDY: "On the representation of numbers as sums of squares and in particular of five and seven."

(5) Dr. J. W. ALEXANDER: "On the representation of any n -dimensional two-sided manifold as a generalized Riemann surface."

(6) Dr. J. W. ALEXANDER: "On the equilibrium of a fluid mass at rest."

(7) Dr. T. H. GRONWALL: "Qualitative properties of the ballistic trajectory (second paper)."

(8) Dr. T. H. GRONWALL: "On the distortion in conformal mapping."

(9) Professor A. A. BENNETT: "Fictitious matric roots of the characteristic equation."

(10) Professor PIERRE BOUTROUX: "On multiform functions defined by differential equations of the first order."

(11) Professor B. H. CAMP: "The significance of a difference, and the value of a sample."

(12) Professor J. H. M. WEDDERBURN: "On division algebras."

(13) Professor EDWARD KASNER: "Geodesics of surfaces and higher manifolds."

Professor Boutroux's paper was a special expository presentation prepared at the suggestion of the programme committee. In the absence of the authors the papers of Professor Lipka, Dr. Schweitzer, Mr. Hardy, and Dr. Alexander's first paper were read by title. Abstracts of the papers follow below.

1. At the annual meeting of the Society, Professor Lipka presented the theorem: if a system of $\infty^{2(n-1)}$ trajectories in space of n dimensions is such that any ∞^{n-1} trajectories of the system which meet an arbitrary hypersurface (space of $n-1$ dimensions) orthogonally admit of ∞^1 orthogonal hypersurfaces, then the system may be considered as the

trajectories in a conservative field of force with a given constant of energy. The purely geometric part of the theorem is true for all natural families of curves. The n -dimensional spaces considered were spaces of constant curvature. In the present paper, the above theorem is demonstrated for spaces of variable curvature. The result is the converse of the Lipschitz theorem announced in *Crelle's Journal*, volume 74 (1871).

2. Given a set S consisting of at least two elements, Dr. Schweitzer shows that this set constitutes an algebra of logic provided the following postulates are satisfied in addition to the property of unique closure with regard to the undefined relations $\theta(x, y)$ between the elements x, y of S :

$$1. \theta(x, x) = \theta(y, y). \qquad 2. \theta\{x, \theta(y, x)\} = x.$$

3. There exists an element 0 such that for every x, y, z , $\theta\{x, \theta(y, z)\} = \psi\theta\{\psi\theta(x, y), \theta[z, \psi(x)]\}$, where $\psi(t) = \theta[0, t]$.

In the second part of the paper iterative relations based on the preceding postulates are given; these relations are mainly in generalization of the relations

$$\theta\{\theta(x, y), \theta(y, z)\} = \theta(x, y),$$

$$\theta\{\theta(y, x), \theta(z, y)\} = \theta(y, x).$$

Concretely, $\theta(x, y)$ may be interpreted by $x + \text{not-}y$.

3. A formal (material) pseudogroup is any system of elements E satisfying the properties $S[\rho_1, \rho_2, \dots]$ necessary for a formal (material) group and including closure with reference to the undefined relations ρ_i generating S , apart from a set E_0 of exceptional elements. If the set E_0 is empty, then the pseudogroup is proper; otherwise, improper. Dr. Schweitzer proves the following theorem: The abstract field is an improper formal pseudogroup $S[\phi, \psi, \phi_1, \psi_1]$ subject to the condition $\psi_1[x, \phi_1(y, x)] = y$. In case of the abstract group the undefined relations have the interpretations

$$\phi(x, y) = x \cdot y, \qquad \phi_1(x, y) = x^{-1}yx,$$

$$\psi(x, y) = x^{-1} \cdot y, \qquad \psi_1(x, y) = xyx^{-1},$$

and in case of the abstract field the relations may be inter-

puted thus:

$$\phi(x, y) = x + y, \quad \phi_1(x, y) = x \cdot y,$$

$$\psi(x, y) = -x + y, \quad \psi_1(x, y) = \frac{1}{x} \cdot y.$$

This result is analogous to a result obtained previously by the author for the algebra of logic.

4. Mr. Hardy's paper contains a detailed development of certain researches of which a short account has been published in volume 4, pages 189-193, of the *Proceedings of the National Academy of Sciences*. The paper will appear in the *Transactions* of the Society.

5. Dr. Alexander proves that any two-sided manifold of n -dimensions may be represented as a generalized Riemann surface spread over a hypersphere.

6. The question was raised by Liapounoff and Poincaré as to whether there exist any figures of equilibrium besides the sphere for a homogeneous fluid mass. In this paper, Dr. Alexander shows that there are no such figures whether of stable or unstable equilibrium.

7. The differential equations of the trajectory being

$$x'' = -G(v)H(y)x', \quad y'' = -G(v)H(y)y' - g,$$

Dr. Gronwall shows that under fairly broad assumptions on the mode of increase of the functions G and H , the velocity v has only a finite number of maxima and minima as the time t varies from zero to infinity.

8. In the *Comptes rendus de l'Académie des Sciences de Paris*, February 28, 1916, Dr. Gronwall investigated the upper and lower bounds of $|w|$ and $|dw/dr|$, where

$$w = z + a_2 z^2 + \dots + a_n z^n + \dots$$

maps the circle $|z| < 1$ conformally on a simply connected and nowhere overlapping region in the w -plane, and the coefficient $a_2 = ae^{\gamma i}$ ($a > 0$, γ real) is given a priori. He obtained the exact values of these bounds, except the upper

bound of $|z|$ in the case where $0 \leq a < 1$, this case being inaccessible to the method used.

In the present paper, it is shown that for any r between zero and unity, and for $0 \leq a < 1$, we have for $|z| = r$

$$|w| < \frac{1}{4} \left\{ \frac{1}{1-a} \left[1 - \left(\frac{1-r}{1+r} \right)^{1-a} \right] - \frac{1}{1+a} \left[1 - \left(\frac{1-r}{1+r} \right)^{-1-a} \right] \right\},$$

except for the function w obtained by replacing r by $ze^{-\gamma i}$ in the expression to the right, the upper bound being then attained for $z = re^{\gamma i}$.

9. In this paper Professor Bennett shows that many of the simple relations connected with the characteristic equation of a square matrix of order n become obvious if there be assumed to exist n formal matrix quantities (one of which is the given matrix) which are roots of the characteristic equation, commutative with the given matrix, and for which the elementary symmetric functions are scalars. These formal matrices do not always have an actual existence, and their use corresponds in a sense to the introduction of imaginary numbers into algebra.

10. Professor Boutroux's paper first gives a classification of the different types of differential equations belonging to the family $y' + A_0 + A_1 y^2 + A_2 y^3 = 0$, where the A 's are polynomials in x . It then solves, for the simplest type, a problem which has been briefly characterized in the abstract of a paper presented (by title) at the Saint Louis meeting of the Society (see March BULLETIN, pages 271-272).

11. The usual criterion of Tchebycheff for evaluating the probability of the existence of a datum further from the mean value of a distribution than a given multiple of the standard deviation is known to be inadequate. Closer tests are derived by Professor Camp, and applied to the problems of finding the significance of a difference and the value of the mean of a sample. New inequalities used in connection with the point binomial permit a rigorous demonstration of the theorem that, when the distribution from which the

samples are drawn is a point binomial, the distribution of the means may be considered to be the Gaussian law without appreciable error. The point binomial is found to have an invariant property similar to that enjoyed by the Gaussian law.

12. In this paper, Professor Wedderburn shows that the only non-commutative division algebras of order 9 are of the type discovered by L. E. Dickson.

13. Professor Kasner discusses first the ∞^1 geodesics of a surface passing through a given point, giving theorems about the distribution of the osculating spheres. He then studies the ∞^1 plane curves obtained by projecting the geodesics on the tangent plane. These curves all have zero curvature; but J , the rate of variation of curvature with respect to arc, is shown to be a cubic function of the slope. The reconstruction of the surface from the ∞^1 plane curves is then discussed.

For a curved three-dimensional manifold we have ∞^2 geodesics at a point. These are projected on the tangent three-flat, thus giving a bundle of curves in ordinary space. It is shown that these curves have inflexions, the torsion vanishes, but a cubic law is obtained for J . To each tangent line there corresponds an osculating plane; this gives rise to a certain integration problem, and integral cones of the form $X^\alpha Y^\beta Z^\gamma = \text{constant}$ are produced, which are recognized as W -cones (analogous to W or anharmonic curves). The correspondence between tangent and binormal for the curves of the bundle is found to be a quadratic Cremona transformation, and is simply visualized and determined by the indicatric quadric surface of the given manifold at the given point. Generalizations for four-dimensional manifolds (required in Einstein's theory) are stated.

Finally the author extends to general manifolds his theorems concerning the limit of the arc to the chord given in this BULLETIN, volume 20 (1914), pages 524-531. This is of importance because when the minimal lines are real, as in the Einstein theory of gravitation, where they appear as light rays, the peculiar limits obtained instead of unity, namely 0.94, 0.86, 0.80, etc., have physical significance.

F. N. COLE,
Secre ary.

SOME RECENT DEVELOPMENTS IN THE
CALCULUS OF VARIATIONS.*

BY PROFESSOR GILBERT AMES BLISS.

It is my purpose to speak this afternoon of a part of the theory of the calculus of variations which has aroused the interest and taxed the ingenuity of a sequence of mathematicians beginning with Legendre, and extending by way of Jacobi, Clebsch, Weierstrass, and a numerous array of others, to the present time. The literature of the subject is very large and is still growing. I was discussing recently the title of this address with a fellow mathematician who remarked that he was not aware that there had been any recent progress in the calculus of variations. This was a very natural suspicion, I think, in view of the fact that the attention of most mathematicians of the present time seems irresistibly attracted to such subjects as integral equations and their generalizations, the theory of definite integration, and the theory of functions of lines. It is indeed in these latter domains that the activities especially characteristic of the present era are centered, and the progress already made in them, and the further progress inevitable in the near future, will doubtless be sufficient alone to insure for our generation of mathematical workers a noteworthy place in the history of the science.

While speaking of present day mathematical tendencies I should like to take occasion to mention a remark which has been made to me a number of times by persons who are interested in mathematics primarily for its applications. The feeling of some of these scientists seems to be that mathematical research in America is drawing farther and farther away from the forms of mathematics most immediately useful in related subjects, and they wish to attract the attention and stimulate the interest of mathematicians in the directions of these intermediate domains. We should indeed sympathize heartily with this desire. It is not at all to be regarded

* Address of the retiring chairman of the Chicago Section of the American Mathematical Society, read at the joint meeting of the Chicago and Southwestern Sections, the Missouri Section of the Mathematical Association of America, and Section A of the American Association for the Advancement of Science, at St. Louis, December 30, 1919.

as unfortunate that so much research is being carried on in this country in the purely mathematical field, for one must not forget that the history of mathematics presents repeated instances of pure mathematics of former times which has become applied mathematics of to-day. Furthermore from the educational standpoint it is most important to note that the spirit of research itself, apart from the results attained, is the guarantee in any individual of continuous self-instruction and development, and consequent increase in value to his community. It does seem unfortunate, however, that for some reason the number of people in this country interested in the adaptation of mathematical theories to the pressing contemporary problems of the applications has been relatively small. I am hoping sincerely that cooperation of mathematicians with scientists in other fields will presently be effective in remedying this defect. The mathematical community should assist in every possible way in developing strong applied mathematical research centers especially where beginnings have already been made, in encouraging graduate students to investigations in the fields intermediate between mathematics and the neighboring sciences, and in the discovery and dissemination among maturer workers of information concerning the applied mathematical problems now awaiting solution. In this connection let me also suggest that mathematicians would do well to inquire searchingly into the advantages or disadvantages of cooperative work among themselves, as well as with scientists in other fields. Cooperation seems to be the watchword of the time, and we should not be the last to recognize it. I am myself not yet convinced of the economy of group effort as over against individual effort in mathematical research, but I believe it important that in the near future we should have some conclusive information about it.

Let me return, however, to the original topic of my discourse which was to have been a part of the theory of the calculus of variations. I should like very much to dwell on some of the modifications which have been made possible by recent progress in the domains of real function theory mentioned in my first paragraph above, but it would be impossible in the limited time at my disposal. Instead I wish to speak of a new method of treating the second variation in the calculus of variations, a method which has rested partly developed

in my mind for some time, and which has recently turned out to be effective in simplifying the theory of the second variation and coordinating the complicated and extensive literature of the subject.

The simplest problem of the calculus of variations, the one to which my expository remarks this afternoon must be for the most part restricted, is that of determining a curve

$$(C) \quad y = y(x) \quad (x_1 \leq x \leq x_2)$$

in the xy -plane, joining two fixed points (x_1, y_1) and (x_2, y_2) , and minimizing an integral

$$(1) \quad J(C) = \int_{x_1}^{x_2} f(x, y(x), y'(x)) dx.$$

In order to obtain conditions which must be satisfied by a minimizing arc let us consider the values of the integral along the curves of a family of the form

$$\bar{y} = y(x) + \alpha \eta(x) \quad (x_1 \leq x \leq x_2),$$

where α is a constant to be varied at pleasure and $\eta(x)$ is a function which vanishes at x_1 and x_2 , so that $\eta(x_1) = \eta(x_2) = 0$. All of the curves of this family pass through the end points of the original curve C , on account of the last condition imposed upon the $\eta(x)$, and it is clear that the function of α ,

$$J(\alpha) = \int_{x_1}^{x_2} f(x, y + \alpha \eta, y' + \alpha \eta') dx,$$

must have a minimum for $\alpha = 0$, so that by the usual theory of maxima and minima the conditions

$$(2) \quad J'(0) = \int_{x_1}^{x_2} (f_y \eta + f_{y'} \eta') dx = 0,$$

$$(3) \quad \begin{aligned} J''(0) &= \int_{x_1}^{x_2} (f_{yy} \eta^2 + 2f_{yy'} \eta \eta' + f_{y'y'} \eta'^2) dx \\ &= \int_{x_1}^{x_2} 2\Omega(x, \eta, \eta') dx \geq 0 \end{aligned}$$

must be satisfied for every choice of the function $\eta(x)$ vanishing at x_1 and x_2 . Here the arguments of the derivatives of f are the $y(x)$ and $y'(x)$ which belong to the curve C , and the symbol

$2\Omega(x, \eta, \eta')$ is merely a notation for the quadratic form which appears in the integrand of $J''(0)$. The expressions (2) and (3) are called, respectively, the first and second variations of the integral J along the arc C .

The first term of the first variation can be integrated by parts in simple fashion giving the equation

$$J'(0) = \int_{x_1}^{x_2} \left(f_{y'} - \int_{x_1}^x f_y dx \right) \eta' dx = 0$$

which must hold for every $\eta(x)$ vanishing at x_1 and x_2 , and it is provable from this condition that there must be a constant c such that along the minimizing arc C

$$(4) \quad f_{y'} = \int_{x_1}^x f_y dx + c.$$

This is a very important equation from which can be deduced several consequences. In the first place the derivative $f_{y'}$ must be continuous all along the minimizing arc C , since the second member of the equation has this property, so that at each corner point (x_0, y_0) of C the equation

$$(5) \quad f_{y'}(x_0, y_0, y'(x_0 - 0)) = f_{y'}(x_0, y_0, y'(x_0 + 0))$$

must be satisfied by the two slopes $y'(x_0 - 0)$, $y'(x_0 + 0)$ of the arc C toward the right and left from the corner. In the second place it is provable at once by differentiating the two sides of the equation that the differential equation

$$(6) \quad \frac{d}{dx} f_{y'} = f_y$$

must be satisfied everywhere on C . Finally it follows from the equation (4) by somewhat more complicated reasoning that the minimizing arc, supposed at first to have only a first derivative y' elsewhere than at corners, must have also higher derivatives, at least at every point at which the condition $f_{y'y'} \neq 0$ is satisfied, i.e. at every point at which the differential equation (6) is non-singular.

The differential equation (6) of the minimizing arc is now usually ascribed to Euler,* though for a long time it was called

* *Methodus inveniendi*, etc. (1744), Chapter II, Section 21; translated into German in Ostwald's *Klassiker der exakten Wissenschaften*, No. 46, p. 53.

the Lagrange equation, because Lagrange adopted it and used it so widely. The integration by parts which I have mentioned in treating the first variation is due to du Bois Reymond.* The corner condition (5) was deduced by Weierstrass for the parametric case in his lectures as early as 1865, but was re-discovered by Erdmann in 1877† before the results of Weierstrass were published. The differentiability properties of the minimizing arc were studied by Hilbert‡ by means of the equation (4), but I think that the derivation of them from this equation by a very simple application of implicit function theorems was due to Mason and myself.§

Before proceeding to the theory of the second variation which is my ultimate goal, I must mention two further contributions to the theory by Weierstrass, familiar to those of you who have studied the calculus of variations, but perhaps not so to others. In his lectures of 1879 Weierstrass introduced his so-called *E*-function

$$E(x, y, y', \bar{y}') = f(x, y, \bar{y}') - f(x, y, y') - (\bar{y}' - y')f_{y'}(x, y, y').$$

It is clearly the difference of $f(x, y, \bar{y}')$ and the sum of the first two terms obtained by expanding this function by Taylor's formula in powers of $\bar{y}' - y'$. With the help of very simple types of comparison curves he established the further condition that

$$E(x, y, y', \bar{y}') \geq 0$$

at every set of elements $(x, y, y', \bar{y}')$ for which x, y, y' belongs to the arc C and $\bar{y}'$ is arbitrary. This was a new and important result, for all the necessary conditions deduced before his time had involved only comparison curves which hugged closely the original arc C both in position and direction, and he was the first to show that the admission of variations which departed widely in direction, though not in position, from the minimizing arc C gives rise to this further restrictive property.

But he added to the theory in other important respects also, only one of which I can here lay before you. The equation (6), after the differentiation with respect to x has

* *Math. Annalen*, vol. 15 (1879), p. 313.

† *Journal für Mathematik*, vol. 82 (1877), p. 21.

‡ *Göttinger Nachrichten*, 1900, p. 253.

§ "The properties of curves in space which minimize a definite integral," *Trans. Amer. Math. Society*, vol. 9 (1908), p. 440.

been performed, has the form

$$f_{v'x} + f_{v'y}y' + f_{v'y'}y'' = f_v,$$

which shows it to be of the second order, and its solutions form a two-parameter family of curves which are called the extremals of the problem. Weierstrass considered especially one-parameter families of such extremals of the form

$$(7) \quad y = \varphi(x, a),$$

containing the particular arc C , whose minimizing properties are under investigation, for a particular parameter value a_0 . If a family of this sort can be found which simply covers a neighborhood F of C , then the region F with its covering of extremals is called a field about C . By saying that F is simply covered by the family (7) it is meant that through each point of the region there passes one and but one extremal of the family, so that with each point (x, y) of the field F there is associated a unique extremal slope $p(x, y)$. If now $\bar{C}$ is an arbitrary curve

$$(\bar{C}) \quad y = \bar{y}(x) \quad (x_1 \leq x \leq x_2)$$

in the field F joining the end points of C then Weierstrass showed that the difference $J(\bar{C}) - J(C)$ of the values of J along the two arcs is expressible by the formula

$$(8) \quad J(\bar{C}) - J(C) = \int_{x_1}^{x_2} E(x, \bar{y}, p(x, \bar{y}), \bar{y}') dx$$

in which $\bar{y}$ stands always for the function $\bar{y}(x)$ defining the arc $\bar{C}$.

It is clear from this formula that $J(C)$ will surely be a minimum if the arc C can be imbedded in a field F of the type described above, and if the integrand in the second member of the formula (8) is always positive along arcs of the type of $\bar{C}$ in the field. I should be giving a resumé of most of the theory of the calculus of variations if I should attempt to describe in detail the conditions under which the field F can be constructed, and which insure the minimizing properties of C . They are, however, well known, and for this simple case are not difficult to develop and present.

Hilbert* devised a very elegant method of establishing the formula (8) which has now superseded with most writers the original "Weierstrass construction" by means of which Weierstrass himself established his result. Hilbert showed that the value of the integral

$$J^*(\bar{C}) = \int_{x_1}^{x_2} \{f(x, \bar{y}, \bar{y}') + [\bar{y}' - p(x, \bar{y})]f_{y'}(x, \bar{y}, p(x, \bar{y}))\} dx$$

is independent of the path $\bar{C}$ chosen in the field joining two fixed points, and that along an extremal arc of the field $J^*(C) = J(C)$, since along such an arc the differential equation $y' = p(x, y)$ is always satisfied. It follows then at once that

$$\begin{aligned} J(\bar{C}) - J(C) &= J(\bar{C}) - J^*(C) = J(\bar{C}) - J^*(\bar{C}) \\ &= \int_{x_1}^{x_2} E(x, \bar{y}, p(x, \bar{y}), \bar{y}') dx. \end{aligned}$$

From a very early period in the history of the calculus of variations to the time of Weierstrass it was thought that an arc C which made the first variation zero and the second variation positive for all choices of the function η , would surely minimize the integral J . The acceptance of this assumption, unjustified by any proof, was due to the analogy between the calculus of variations and the theory of maxima and minima of ordinary functions of ordinary variables. The assumption is correct in the case of a weak minimum, for which the only comparison curves admitted are those which are near to C both in position and in direction, but Weierstrass showed that it is incorrect in the case of a strong minimum, for which the comparison curves are required to be close to C in position only. He developed his necessary E -function condition, in addition to those which are deduced from the first and second variations, and proved that the conditions so found when slightly modified are sufficient to insure the existence of a minimum along the arc C .

One result of this partial misconception by those who preceded Weierstrass was an elaborate theory of the second variation, the purpose of which was to reduce it to a form in which it would be clearly positive for all choices of the function

* An exposition is given by Bolza, *Vorlesungen über Variationsrechnung*, pp. 105ff.

$\eta(x)$. Legendre, in 1786,* was the first to attempt such a transformation. Euler, Lagrange, and other investigators before him had obtained from the first variation the differential equation of the minimizing curves, and conditions at the ends of the curves when the end points were allowed to vary. He himself begins his paper rather apologetically, it seems to me, his avowed purpose being to exhibit a criterion for distinguishing between extremal arcs which furnish maxima and those which furnish minima, whereas he says that in a majority of the applications one can usually readily decide this question from the setting of the problem itself. Inspired, however, by the analogous theory of maxima and minima of functions of a finite number of variables, he succeeded in transforming the second variation into the form

$$J''(0) = \int_{x_1}^{x_2} f_{y'y'} \left(\eta' + \frac{f_{yy'} + \alpha}{f_{y'y'}} \eta \right)^2 dx,$$

where α is a solution of a certain Riccati differential equation of the first order. From this he concluded that the second variation will always be positive and therefore that the integral (1) will always be minimized by an extremal arc C along which $f_{y'y'} > 0$. His proof was faulty because there does not always exist a function α of the type he used, finite everywhere between the limits x_1 and x_2 , but his conclusion that the positive form found for the second variation would ensure a minimum is correct for the case of a weak minimum, as I have already indicated. It turns out that the condition, $f_{y'y'} \geq 0$ along the minimizing extremal arc, is necessary for a minimum but not always sufficient, and it is now called the Legendre necessary condition. It is interesting to note that Legendre himself was not entirely satisfied with his analysis, for he published later comments on his own paper in which he called attention to some of the weaknesses of his argument and suggested methods of overcoming them.

In 1837 Jacobi† announced the discovery of an important advance in the theory of the transformation of the second variation. He disclosed the circumstances under which the trans-

* "Abhandlung über die Untersuchung der Maxima und Minima in der Variations-Rechnung," *Mémoires de l'Académie royale des Sciences*, 1786, p. 7.

† "Zur Theorie der Variationsrechnung und der Differentialgleichungen," *Journal für die reine und angewandte Mathematik*, vol. 17 (1837), p. 68.

formation of Legendre was possible and characterized the cases when it is not possible. On every extremal arc with a fixed initial point A there is in general a point A' conjugate to A such that Legendre's transformation is possible on every arc AB excluding A' , but not possible on an arc AB containing A' . In justifying these statements Jacobi introduced, in place of the Riccati equation used by Legendre, a linear differential equation of the second order, now usually called the Jacobi differential equation, and he showed that its solutions can be obtained from those of Euler's equation by differentiations with respect to the constants of integration. He also used a new method for transforming the second variation which he applied successfully to the case when the integrand of the integral (1) contains higher derivatives of y . His conclusions were that an extremal arc AB which contains the conjugate point A' either between A and B or at B can not minimize the integral J , and that an extremal arc AB which has $f_{y'y'} > 0$ along it and which does not contain the conjugate point A' will surely minimize J .

Jacobi's results were correct with the exception of his statement that no minimum can exist when the point A' conjugate to A coincides with B . It is now known that in this exceptional case there will sometimes be a minimum and sometimes not. But his proofs were again incomplete. I do not find in examining his paper that he justifies anywhere the necessity of the condition that the conjugate point A' shall not lie between A and B , and his sufficiency proof rests solely upon his transformation of the second variation to positive form and is open to the same objection which I have mentioned before as applying to all of the writers who preceded Weierstrass.

In extending his transformation to integrals containing higher derivatives of the function $y(x)$ Jacobi made use of a theorem concerning linear differential equations which he did not prove. It is closely associated with the now well-known relation

$$u J(v) - v J(u) = \frac{d}{dx} M(u, v)$$

satisfied by a self-adjoint linear differential equation of the n th order. His incomplete exposition aroused much interest and led to a series of papers by Delaunay, Minding, Spitzer,

Heine, Clebsch, Hesse, Lipschitz, Mayer, Scheffers, von Escherich, and others, who gave proofs of the Jacobi theorem and extended his results to more complicated problems of the calculus of variations. Of these papers I shall mention especially only those which seem to me to contain the more important advances.

In 1858 Clebsch* studied the transformation of the second variation for the very general type of problem formulated by Lagrange. For such a problem there is an integral of the form

$$\int_{x_1}^{x_2} f(x, y_1, \dots, y_n, y_1', \dots, y_n') dx,$$

to be minimized in the class of curves

$$y_i = y_i(x) \quad (x_1 \leq x \leq x_2; i = 1, \dots, n)$$

which join two fixed points and which satisfy besides a set of differential equations

$$\varphi_i(x, y_1, \dots, y_n, y_1', \dots, y_n') = 0 \quad (i = 1, \dots, m < n).$$

Clebsch found the system of linear differential equations of the second order analogous to the single Jacobi equation in the simpler case which we have been considering, and he expressed the second variation as an integral

$$(9) \quad \int_{x_1}^{x_2} \sum_{i,k} F_{y_i' y_k'} W_i W_k dx.$$

In this expression F is the sum

$$F = f + \lambda_1 \varphi_1 + \dots + \lambda_m \varphi_m,$$

the coefficients λ_i being functions of x , and the variables W_i are proportional to certain determinants of order $n + 1$ which satisfy the relations

$$\frac{\partial \varphi_i}{\partial y_1'} W_1 + \dots + \frac{\partial \varphi_i}{\partial y_n'} W_n = 0 \quad (i = 1, \dots, m).$$

The Jacobi transformation of the second variation for the simpler problems is possible if there is a solution u of

* *Journal für die reine und angewandte Mathematik*, vol. 55 (1858), pp. 254 and 335; in two papers: "Ueber die Reduktion der zweiten Variation auf ihre einfachste Form," "Ueber diejenigen Probleme der Variationsrechnung, welche nur eine unabhängige Variable enthalten."

Jacobi's equation which does not vanish on the interval x_1x_2 . For the Lagrange problem here under consideration there must be an n -square matrix with columns belonging to n so-called conjugate systems of solutions of the Jacobi equations and having a determinant different from zero on x_1x_2 .

The transformation of Clebsch seems to me a very complicated one. It is based upon a clever foresight of the form to which the integrand of the second variation may be reduced, and involves a count of the conditions necessary to make the reduction, the introduction of a sufficient number of arbitrary functions to satisfy these conditions, and the determination of the functions so introduced by a sequence of systems of equations. From the results of the transformation one may infer that the second variation will be positive whenever the quadratic form in the integrand of the expression (9) is positive at each point of the extremal arc under consideration. It is also provable that this property is necessary for a minimum, and the condition so found is sometimes called the Legendre-Clebsch condition because it is the generalization of the one which Legendre deduced for the simpler case.

Ten years after the papers of Clebsch appeared A. Mayer* presented the same results in somewhat simpler form and extended the Jacobi theory of the conjugate point to fit the Lagrange problem. He determined the position of the conjugate point by means of a zero of a determinant of solutions of Jacobi's equations, now called the Mayer determinant.

But the most thorough study of the theory of the transformation of the second variation is undoubtedly that of von Escherich.† In a series of five papers in 1898, 1899, and 1901 he effected the Clebsch transformation for several types of problems by a generalization of Jacobi's method, and studied in great detail the theory of conjugate points and the conjugate systems of solutions of Jacobi's equations which are important in the determination of these points as well as in the transformation itself. The equations which I am designating here as the Jacobi equations, because they are generalizations of

* "Ueber die Kriterien des Maximums und Minimums der einfachen Integrale," *Journal für die reine und angewandte Mathematik*, vol. 69 (1868), p. 238.

† "Die zweite Variation der einfachen Integrale," *Sitzungsberichte der kaiserlichen Akademie der Wissenschaften in Wien, Mathematisch-naturwissenschaftliche Classe*, vols. 107, 108, 110 (1898, 1899, 1901).

the one which Jacobi used in the simpler case, are called by von Escherich the accessory system of linear differential equations. I will mention especially, without attempting to write it down, an important formula which lies at the basis of the von Escherich theory. It is a formula involving the determinants of conjugate systems and is of great assistance in the discussion of the zeros of the Mayer determinant which determines conjugate points. It should be added that von Escherich discovered, as a result of his detailed study, an important exceptional case for the Lagrange problem which he distinguishes from the more general normal or non-singular type. Most of his results apply only to the latter, and little is in fact known of the possibilities of the singular case.

A discussion of the second variation could not be complete without mentioning the work of Weierstrass. His results were obtained between 1865 and 1888 but they became known very slowly because he described them only in his lectures. The book by Kneser, *Vorlesungen über Variationsrechnung*, 1900, was I think the first published account of any importance. Weierstrass confined his attention for the most part to the problem in parametric form in the plane. His transformation of the second variation for this case was exceedingly ingenious, but apparently not readily generalizable to higher spaces. His other work, however, had also an important influence upon the study of the second variation. For the sufficiency proof which he developed for the case of a strong minimum applies also to a weak minimum, and a sufficiency proof in the latter case based on the theory of the transformation of the second variation is therefore superfluous.

Finally I should mention an interesting paper by Hahn* in which he shows the connection between the conjugate systems of solutions of Jacobi's equations and what are called Mayer fields. These latter are generalizations of the plane fields of which I have spoken above. For problems in higher spaces the analogue of the field covered by a one-parameter family of solutions of Euler's equations is an n -parameter family of extremals simply covering a region F of space. But the Hilbert integral will be invariant in such a region F only if the n -parameter family itself has special properties, in which case F with its extremals is called a Mayer field

* "Ueber den Zusammenhang zwischen den Theorien der zweiten Variation und der Weierstrass'schen Theorie der Variationsrechnung," *Rendiconti del Circolo Matematico di Palermo*, vol. 29 (1910), p. 49.

because Mayer was the first to recognize and study the properties of such families. The phenomena presented by the n -parameter families of Mayer fields are analogous to those of two-parameter families of straight lines in space. Not every such family is orthogonal to a surface. When one has this property it will form a Mayer field for the problem of minimizing the ordinary length integral in space. It turns out that every conjugate system of solutions of the Jacobi equations is deducible from the equations of the n -parameter family of a Mayer field by differentiation with respect to its parameters, and Hahn uses this property to assist in constructing the various types of conjugate systems which von Escherich studied.

I have now come to the results of which I wish especially to speak this afternoon. The historical sketch which I have given, though most incomplete, will suffice to show the great interest which mathematicians have taken in the theory of the second variation, and the complicated character of the transformations which have been devised for it. The results of Weierstrass mentioned above, and some very beautiful geometrical methods, developed by Darboux for geodesics and generalized by Kneser, enable one to deduce the necessary conditions of Legendre and Jacobi in some cases without the use of the complicated theory of the second variation. These proofs fail, however, when the enveloping curve upon which they depend has a singularity of a special type, and the efforts which have been made to complete the proofs to cover all cases have led to complications as great as those of the theory of the second variation itself.

The importance of the theory of the second variation at the present time lies therefore primarily in the proofs which it provides for the necessity of the analogues, for more general problems, of the conditions discovered by Legendre and Jacobi for the simpler cases. Furthermore the transformation of the second variation into a form clearly positive may be made the basis for the justification of a set of sufficient conditions for a weak minimum, and if the transformation can be effected simply the method will be an economical one. Finally I regard it as highly desirable from a historical point of view that the complicated theories which have been attached to the second variation should have an interpretation, in the light of recent results in the calculus of variations, which will make them more

perspicuous and less detached than they seem to be at present from the theory as a whole.

For some years past I have been in the habit, in my lecture courses at the University of Chicago, of deducing the Legendre and Jacobi conditions for simpler problems by an application to the second variation of results described in the earlier part of this paper, the first two necessary conditions mentioned as deducible from the first variation and the Weierstrass E -function condition. The underlying thought is this. Since the second variation must always be positive or zero along a minimizing arc for all choices of the function $\eta(x)$ such that $\eta(x_1) = \eta(x_2) = 0$, we are led to the consideration of a minimum problem in the $x\eta$ -space for the integral (3) of the second variation. This problem is of precisely the same sort as the original problem of minimizing the integral (1). The curve $\eta = 0$ is necessarily a minimizing curve for the $x\eta$ -problem, when $J(C)$ is a minimum, since it gives the second variation its smallest value zero. Hence along it we must have the E -function condition satisfied,

$$\begin{aligned}\Omega(x, \eta, \bar{\eta}') - \Omega(x, \eta, \eta') - (\bar{\eta}' - \eta')\Omega_{\eta'}(x, \eta, \eta') \\ = f_{v'v'}(\bar{\eta}' - \eta')^2 \geq 0,\end{aligned}$$

the first member of the equality being the Weierstrass E -function for the integrand function Ω of the second variation. It follows at once that the Legendre condition $f_{v'v'} \geq 0$ must hold along the minimizing arc.

Furthermore, the Euler differential equation of this minimum problem of the second variation is the equation

$$\Omega_{\eta} - \frac{d}{dx}\Omega_{\eta'} = (f_{vv}\eta + f_{vv'}\eta') - \frac{d}{dx}(f_{v'v}\eta + f_{v'v'}\eta') = 0$$

which is precisely the linear differential equation of the second order in η upon which Jacobi based his theory of the second variation. The coefficient of η'' in this equation is $f_{v'v'}$, and in discussing the Jacobi condition it is customary to presuppose this coefficient different from zero along the arc C , so that the solutions of the Jacobi equation will have no singularities on the interval x_1x_2 . Suppose now that there is a solution u of Jacobi's equation not identically zero but vanishing at x_1 and at a second point x_3 between x_1 and x_2 . Then in the first place the value $u'(x_3)$ is different from zero, since the

only solution of Jacobi's equation vanishing with its derivative at any point whatsoever is $\eta \equiv 0$. Furthermore the function $\eta(x)$ defined by the conditions

$$\begin{aligned}\eta(x) &= u(x) & \text{for } x_1 \leq x \leq x_3, \\ \eta(x) &= 0 & \text{for } x_3 \leq x \leq x_2,\end{aligned}$$

is an admissible function η . It is easily provable that $\eta(x)$ gives the second variation its minimum value zero, for with the help of an integration by parts and properties of the homogeneous quadratic function Ω we have

$$\begin{aligned}J''(0) &= 2 \int_{x_1}^{x_3} \Omega(x, u, u') dx = \int_{x_1}^{x_3} (u\Omega_u + u'\Omega_{u'}) dx \\ &= u\Omega_{u'} \Big|_{x_1}^{x_3} + \int_{x_1}^{x_3} u \left(\Omega_u - \frac{d}{dx} \Omega_{u'} \right) dx = 0.\end{aligned}$$

The curve defined by $\eta(x)$ can not, however, be a minimizing curve for the second variation since it has a corner at x_3 at which

$$\Omega_{\eta'}(x_3 - 0) - \Omega_{\eta'}(x_3 + 0) = f_{y'y'} u' \Big|_{x_3} \neq 0.$$

Since $\eta(x)$ gives the second variation the value zero and does not minimize it, we conclude that there are variations $\bar{\eta}(x)$ giving the second variation a value less than zero, and it follows at once that the original integral J can not be minimized by the arc C . We have then the result that if $J(C)$ is a minimum there is no solution $u \neq 0$ of Jacobi's equation vanishing at x_1 and at a second point x_3 between x_1 and x_2 . This is, however, exactly Jacobi's necessary condition, since the conjugate point to $A(x = x_1)$ is the point $A'(x = x_3)$ corresponding to the first zero x_3 of $u(x)$ after x_1 , and we see that arcs AB containing A' between A and B have no longer the minimizing property.

It is to be noted that the proofs of the necessity of the Legendre and Jacobi conditions just described are made under the hypotheses usually presupposed when these conditions are deduced from the second variation, and I wish further to call attention to the fact that they do not involve any transformation of the second variation except the very simple one used in showing that the curve $\eta(x)$ makes the second variation vanish. The Euler differential equation condition, the corner

point property, and the Weierstrass E -function condition, are properties of a minimizing arc which should be developed early in every complete discussion of the problem, and when we apply them to the second variation we are not using new methods peculiar to this later part of the theory. The effectiveness and economy of the methods just described are not as important for the simple case, which has been discussed in some detail, as they are for the more complicated problems of the calculus of variations. D. M. Smith* has, however, extended these methods so that they can be applied to the problem of Lagrange, and Miss G. A. Larew† has done the same for the problem of Mayer. I have myself modified them for problems in parametric form in higher spaces.‡ In these latter cases a transformation of the second variation analogous to that of Weierstrass for the plane seems now entirely unnecessary.

But it has seemed to me for some time that the theory of the second variation in its entirety could be viewed with success from the standpoint of the minimum problem of the second variation, a minimum problem within a minimum problem, so to speak, and I have recently satisfied myself that this is so for the Lagrange problem as well as for the simpler cases. A one-parameter family of extremals of the second variation simply covering a field in the $x\eta$ -plane is a family of solutions of Jacobi's equations of the form

$$(10) \quad \eta = au \quad (x_1 \leq x \leq x_2),$$

where u is a particular solution of the Jacobi equation which does not vanish on the interval x_1x_2 , and a is the parameter of the family. The slope function $\pi(x, \eta)$ for this field, expressing the slope of the extremal of the field through the point (x, η) and corresponding to the slope function $p(x, y)$ of the general case, is found by solving the last equation for a and substituting in the expression

$$\pi(x, \eta) = au' = \eta \frac{u'}{u}.$$

* "Jacobi's condition for the problem of Lagrange in the calculus of variations," *Trans. Amer. Math. Society*, vol. 17 (1916), p. 459.

† "Necessary conditions in the problems of Mayer in the calculus of variations," *Trans. Amer. Math. Society*, vol. 20 (1919), p. 1.

‡ "Jacobi's condition for problems of the calculus of variations in parametric form," *Trans. Amer. Math. Society*, vol. 17 (1916), p. 195.

If $J''(H)$ represents the value of the second variation for an arbitrary function $\eta(x)$ vanishing at x_1 and x_2 , and if $J''(H_0)$ is the corresponding value along $\eta = 0$, then the formula (8) of Weierstrass when applied to the second variation takes the form

$$J''(H) - J''(H_0) = \int_{x_1}^{x_2} \{ \Omega(x, \eta, \eta') - \Omega(x, \eta, \pi) - (\eta' - \pi) \Omega_{\eta'}(x, \eta, \pi) \} dx.$$

But since $J''(H_0) = 0$, an application of Taylor's formula to the integrand on the right gives

$$J''(H) = \int_{x_1}^{x_2} f_{\eta'\eta'} (\eta' - \pi)^2 dx = \int_{x_1}^{x_2} f_{\eta'\eta'} \left(\frac{\eta' u - \eta u'}{u} \right)^2 dx.$$

This is precisely the form of the second variation found by Jacobi. From it we conclude at once that the second variation is positive or zero for every function η vanishing at x_1 and x_2 , provided that $f_{\eta'\eta'} > 0$ along the original arc C , and that a solution u of Jacobi's equation exists which is different from zero on the interval $x_1 x_2$.

The analysis in the case of the Lagrange problem is of course more complicated. The one-parameter family (10) of the field for the second variation is here replaced by an n -parameter family

$$(11) \quad \eta_i = a_1 u_{i1} + \cdots + a_n u_{in} \quad (i = 1, \dots, n),$$

for which the columns of the matrix $||u_{ik}||$ belong to a conjugate system of solutions of the Jacobi equations. The slope functions $\pi_i(x, \eta)$ of the field are found by solving the last equations for the parameters a_i and substituting them in the expressions

$$\pi_i = a_1 u_{i1}' + \cdots + a_n u_{in}' \quad (i = 1, \dots, n),$$

where u_{ik}' is the derivative of u_{ik} with respect to x . The Weierstrass formula gives at once the value

$$J''(H) = \int_{x_1}^{x_2} \sum_{ik} F_{\eta_i' \eta_k'} (\eta_i' - \pi_i) (\eta_k' - \pi_k) dx$$

for the second variation, where F has the significance before explained, and where the differences $\eta_i' - \pi_i$ turn out to

have the values

$$\eta_i' - \pi_i = W_i = \frac{1}{|u_{ik}|} \chi_i(\eta)$$

in terms of the functions W_i of Clebsch and the determinants $\chi_i(\eta)$ of von Escherich. These latter functions have hitherto appeared in somewhat artificial fashion in the theory and it seems to me interesting to have this very simple interpretation of them in terms of the slope functions of the field.

The von Escherich fundamental formula which I mentioned above* may be expressed by the equation

$$\sum_{ik} F_{\eta_i' \eta_k'} (\eta_i' - \pi_i) (\eta_k' - \pi_k) = \sum_i a_i' \psi_i,$$

where the symbols a_i' represent the derivatives of the solutions a_i of the equations (11) with respect to x when the variables η_i are thought of as functions of x , and the coefficients ψ_i are bilinear expressions in the elements of two sets of functions $(\eta_1, \dots, \eta_n; \mu_1, \dots, \mu_m)$ and $(u_{1i}, \dots, u_{ni}; \sigma_{1i}, \dots, \sigma_{mi})$, the latter of which is the system of solutions of Jacobi's equations to which the column $u_{1i}, \dots, u_{ni}$ of the matrix $||u_{ik}||$ belongs. If the set $(\eta; \mu)$ is also a solution of Jacobi's equations then each of these coefficients is a constant. For this special case I have already published a proof† of the von Escherich formula, but the one which I have mentioned here is quite general and simpler than my earlier one.

Finally the von Escherich theory of conjugate systems of solutions of Jacobi's equations is identical with the theory of Mayer fields for the minimum problem of the second variation, and the methods which Hahn devised for relating conjugate systems to the Mayer fields of the original minimum problem can be used to construct the various types of conjugate systems which von Escherich considered.

In conclusion I may say that the methods suggested above seem to me of considerable assistance in developing and understanding the theory of the calculus of variations. For in the first place they enable one to retain the advantages of the second variation, in deducing the necessary conditions analogous to those of Legendre and Jacobi, without becoming involved in elaborate transformations. In the second place

* Von Escherich, loc. cit., Mittheilung IV, Section XXIII, formula (9).

† "A note on the problem of Lagrange in the calculus of variations," this BULLETIN, vol. 22 (1916), p. 220.

they may give some satisfaction to those who have delved in the theory of those transformations, since they make the transformations appear, not as separate chapters of the analysis, but as special applications of formulas now well established in other parts of the theory. It is even possible to make a satisfactory and not over complicated sufficiency proof for a weak minimum, without the use of the Weierstrass notion of a field. For the Weierstrass formula can be proved directly with some ease for the second variation when once it has been seen to hold true for a conjugate system of solutions of Jacobi's equations. It is not strange that the second variation has not been attacked from this standpoint before in spite of the fact that in my recent review of the literature I have found several suggestions which might have instigated one to attempt it. The real reason is, I think, that the advances of Weierstrass and Hilbert were published after 1900 and about the time that Kneser found his envelope theorem. The tendency since then has been to discard the theory of the second variation in favor of the more geometrical theory, but the experiment, so far as I know, has not been completely successful.

THE UNIVERSITY OF CHICAGO,
December, 1919.

GROUPS GENERATED BY TWO OPERATORS OF ORDER THREE WHOSE PRODUCT IS OF ORDER FOUR.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society December 30, 1919.)

§ 1. *Introduction.*

It is known that the only groups which are completely determined by the orders of two generators and the order of their product are the dihedral groups and the groups of movements of the five regular solids known to the ancients. In all other cases two generators which are not restricted except as regards their orders and the order of their product

may give rise to any one of an infinite system of groups.* In particular, there is no upper limit to the number of distinct groups that can be generated by two operators of order three whose product is of order four. The group of smallest order which two such operators can generate is the well known group of order 24 which does not contain any subgroup of order 12, and it is known that this group is defined by the facts that it is generated by two operators of order 3 whose product is of order 4 provided that the square of this product is invariant under each of these two operators of order 3.

In the present article we shall consider the groups generated by s_1 and s_2 when these operators satisfy the following conditions:

$$s_1^3 = s_2^3 = (s_1 s_2)^4 = (s_1^2 s_2)^k = 1 \quad (k = 3, 4, 5).$$

It is unnecessary to consider the case where $k = 2$, since the equations $s_1^3 = s_2^3 = (s_1^2 s_2)^2 = 1$ define the tetrahedral group and hence $s_1 s_2$ could not be of order 4. When $k = 1$ the order of $s_1 s_2$ could evidently not be 4. Hence 3, 4, 5 are the smallest positive integral values of k when $s_1 s_2$ is of order 4.

§ 2. *The Order of $s_1^2 s_2$ is Three.*

From the condition that $(s_1^2 s_2)^3 = 1$ it follows that

$$s_1^2 s_2 s_1^2 = s_2^2 s_1 s_2^2, \quad s_1 s_2^2 s_1 = s_2 s_1^2 s_2.$$

Hence the two operators of order 4, $s_1 s_2$ and $s_2 s_1$, are commutative and generate a group whose order cannot exceed 16. This group must be invariant under the group G generated by s_1 and s_2 since

$$s_1^2 s_2 s_1^2 = s_1^2 s_2^2 \cdot s_2^2 s_1^2 \text{ and } s_2^2 s_1 s_2^2 = s_2^2 s_1^2 \cdot s_1^2 s_2^2.$$

The order of G can therefore not exceed 48. That the order of G cannot be less than 48 when no additional restrictions are imposed on s_1 and s_2 results directly from the fact that the abelian group of order 16 and of type (2, 2) admits an automorphism of order 3 in which no operator except identity corresponds to itself. Hence the theorem:

If two operators s_1, s_2 satisfy the conditions

$$s_1^3 = s_2^3 = (s_1^2 s_2)^3 = (s_1 s_2)^4 = 1$$

but are not otherwise restricted, they generate the group of order

* G. A. Miller, *Amer. Journal of Mathematics*, vol. 24 (1902), p. 96.

48 obtained by extending the abelian group of order 16 and of type (2, 2) by means of an operator of order 3 which transforms this group into itself but is not commutative with any of its operators besides identity.

This group of order 48 evidently contains 32 operators of order 3, 12 operators of order 4, and 3 operators of order 2 in addition to identity. Each operator of order 3 is commutative with its own powers only, while each of the other operators besides identity is commutative with exactly 16 operators of the group.

§ 3. The Order of $s_1^2 s_2$ is Four.

It is easy to see that when $s_1^3 s_2$ is of order 4 the order of $\bar{G}$ is divisible by 168 since the two substitutions $s_1 = abc \cdot def$, $s_2 = aeh \cdot cdg$ satisfy the given conditions, as results from the following equations:

$$s_1 s_2 = abdh \cdot cefg, \quad s_1^2 s_2 = adfh \cdot begc.$$

These two substitutions generate a transitive substitution group of degree seven whose order is a multiple of 168 since $s_1^2 s_2 s_1 s_2^2 = b f e d c g h$. The fact that the order of $s_1^2 s_2 s_1 s_2^2$ is actually 7 when s_1, s_2 represent two abstract operators which satisfy the conditions

$$g_1^3 = g_2^3 = (g_1 g_2)^4 = (g_1^2 g_2)^4 = 1$$

but are not otherwise restricted results from the following equations:

$$\begin{aligned}
& (\delta_1^2 \delta_2 \delta_1 \delta_2)^7 \\
&= \delta_1^2 \delta_2 \delta_1 \delta_2^2 \delta_1^2 \delta_2 \delta_1 \delta_2^2 \delta_1^2 \delta_2 \delta_1 \delta_2^2 \delta_1^2 \delta_2 \delta_1 \delta_2^2 \delta_1^2 \delta_2 \delta_1 \delta_2^2 \delta_1^2 \delta_2 \delta_1 \delta_2^2 \delta_1^2 \delta_2 \delta_1 \delta_2^2 \\
&= \delta_1^2 \delta_2 \delta_1^2 \delta_2 \delta_1 \delta_2 \delta_1 \delta_2^2 \delta_1 \delta_2^2 \delta_1^2 \delta_2^2 \delta_1^2 \delta_2 \delta_1^2 \delta_2 \cdot \\
&\quad \delta_1^2 \delta_2 \delta_1 \delta_2 \delta_1 \delta_2^2 \delta_1 \delta_2^2 \delta_1 \delta_2^2 \delta_1^2 \delta_2^2 \delta_1^2 \delta_2 \delta_1^2 \delta_2 \delta_1 \delta_2^2 \\
&= \delta_2^2 \delta_1 \delta_2^2 \delta_1^2 \delta_2^2 \delta_1 \delta_2 \delta_1^2 \delta_2^2 \delta_1 \delta_2 \delta_1 \delta_2^2 \delta_1^2 \delta_2^2 \\
&= \delta_2^2 \delta_1^2 \delta_2 \delta_1 \delta_2 \delta_1^2 \delta_2 \delta_1^2 \delta_2 \delta_1^2 \delta_2^2 \delta_1^2 \delta_2 \delta_1^2 \delta_2^2 \\
&= \delta_2^2 \delta_1^2 \delta_2 \delta_1^2 \delta_2 \delta_1^2 \delta_2 \delta_1^2 \delta_2 = 1.
\end{aligned}$$

To verify that s_1, s_2 actually generate the simple group of order 168 it may be noted that well-known defining relations of this group are as follows:*

* L. E. Dickson, *Linear Groups*, 1901, p. 303.

$$T^2 = 1, \quad S^7 = 1, \quad (ST)^3 = 1, \quad (S^4T)^4 = 1.$$

Let $T = s_2s_1s_2s_1$ and $S = (s_1^2s_2s_1s_2^2)^2$. Hence $S^4 = s_1^2s_2s_1s_2^2$.

$$(S^4T)^4 = (s_1^2s_2s_1s_2^2s_1)^4 = s_1^2s_2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2s_1 = 1;$$

$$\begin{aligned} (ST)^3 &= s_1^2s_2s_1s_2^2s_1^2s_2s_1^2s_2s_1 \cdot s_1^2s_2s_1s_2^2s_1^2s_2s_1^2s_2s_1 \\ &\quad \cdot s_1^2s_2s_1s_2^2s_1^2s_2s_1^2s_2s_1 \\ &= s_1^2s_2s_1s_2^2s_1^2s_2s_1^2s_2^2s_1^2s_2s_1^2s_2^2s_1^2s_2s_1^2s_2s_1 \\ &= s_1^2s_2s_1s_2s_1s_2^2s_1^2s_2s_1s_2s_1s_2^2s_1^2s_2s_1s_2s_1^2s_1^2 = 1. \end{aligned}$$

It has now been proved that $s_1^2s_2s_1s_2^2$ and $s_2s_1s_2s_1$ generate the simple group of order 168. This is also the group generated by s_1 and s_2 since

$$\begin{aligned} s_1^2s_2s_1s_2^2 \cdot s_2s_1s_2s_1 &= s_1^2s_2s_1^2s_2s_1 = s_2^2s_1s_2^2s_1^2 \\ s_2s_1s_2s_1 \cdot s_1^2s_2s_1s_2^2 &= s_2s_1s_2^2s_1s_2^2 = s_2^2s_1^2s_2s_1^2 \\ s_1s_2s_1^2s_2 \cdot s_2^2s_1^2s_2s_1^2 &= s_1s_2s_1s_2s_1^2 \\ s_1s_2s_1s_2s_1^2 \cdot s_1^2s_2s_1s_2^2 &= s_1s_2s_1s_2s_1s_2s_1s_2 \cdot s_2 = s_2. \end{aligned}$$

Since the group generated by $s_1^2s_2s_1s_2^2$ and $s_2s_1s_2s_1$ contains s_2 it contains also $s_1s_2^2s_1^2$ and $s_1^2s_2s_1^2$. The product of these operators is $s_1s_2^2s_1s_2s_1^2 = (s_2^2s_1^2s_2s_1^2)^{-1} \cdot s_1^2$. This proves that the group generated by $s_1^2s_2s_1s_2^2$ and $s_2s_1s_2s_1$ is identical with the group generated by s_1 and s_2 . Hence the following theorem has been established: *The simple group of order 168 is completely defined by the following equations:*

$$s_1^3 = s_2^3 = (s_1s_2)^4 = (s_1^2s_2)^4 = 1.$$

As is well known, W. Dyck gave in 1882 the following defining equations of this simple group*

$$s_1^7 = 1, \quad s_2^3 = 1, \quad (s_1s_2)^2 = 1, \quad (s_1^5s_2)^4 = 1.$$

§ 4. The Order of $s_1^2s_2$ is Five.

If $s_1 = acb \cdot dfe$ and $s_2 = cdf$ it results that $s_1s_2 = adcb \cdot efd$ and $s_1^2s_2 = abdec$. Hence it follows that s_1, s_2 when subjected to the additional condition that $s_1^2s_2$ is of order 5 generate a group G which has the alternating group of degree 6 for a quotient group. L. E. Dickson gave the following defining relations for this quotient group†:

* *Math. Annalen*, vol. 20, p. 41.

† This BULLETIN, vol. 9 (1903), p. 303.

The transform of this square by s_1 is

$$\begin{aligned}
& s_1^2 s_2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 \\
&= s_1^2 s_2 s_1 s_2^2 s_1^2 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 \\
&= s_1^2 s_2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2^2 s_1^2 s_2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 \\
&= s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 \\
&= s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 \\
&= s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 \\
&= s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2 s_1 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 \\
&= s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1^2 s_2 s_1 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1 s_2 s_1 s_2^2 s_1 s_2^2 \\
&= s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1^2 s_2 s_1 s_2 s_1 s_2^2 s_1^2 s_2 s_1 s_2 s_1 s_2 s_1 s_2^2 s_1 s_2^2 \\
&= s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1^2 s_2 s_1 s_2 s_1 s_2^2 \\
&= s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1^2 s_2 s_1 s_2 s_1 s_2^2 s_1 s_2^2.
\end{aligned}$$

Having proved that this square is invariant under s_1 and s_2 we proceed to prove that the cube of this square is identity:

[illegible]

Since $(t_1^3 t_2 t_1 t_2^2)^2$ is invariant under G and the order of this invariant operator is a divisor of 3, it results that the order g of G must be a divisor of 1,080. If $g = 1,080$, G can be represented as a transitive substitution group of degree 18 since it contains the icosahedral group, as results from the fact that

$$s_1^2 s_2 \quad \text{and} \quad s_1 s_2 s_1 s_2$$

are two operators of orders 5 and 2 respectively whose product is of order 3, since

$$s_1^2 s_2 s_1 s_2 s_1 s_2 s_1^2 s_2 s_1 s_2 s_1^2 s_2 s_1 s_2 s_1 s_2 = s_1 s_2^2 s_1^6 s_2^2 s_1^6 s_2^2 s_1^2 = 1.$$

The subgroup of order 60 composed of all the substitutions which omit one letter of this transitive group is of degree 15 since G contains an invariant operator of order 3, and the six systems of intransitivity of the two invariant substitutions of order 3 of this transitive group are permuted according to the alternating group of degree 6. If this subgroup is transitive, it is completely determined since the icosahedral group can be represented in essentially only one way as a transitive substitution group of degree 15. That is, all such possible representations are conjugate substitution groups. Moreover, this transitive subgroup determines five cycles of the two invariant substitutions of order 3 contained in G .

If we assume that this subgroup is generated by the following substitutions:

$$abc \cdot dkf \cdot enh \cdot goi \cdot jlm, \quad al \cdot bn \cdot cf \cdot do \cdot gm \cdot hi,$$

then the invariant substitutions of the substitution group of degree 18 which is simply isomorphic with G may be assumed to be

$$abc \cdot dmh \cdot ekj \cdot fln \cdot gio \cdot pqr \\ acb \cdot dhm \cdot ekj \cdot fnl \cdot goi \cdot prq.$$

An additional generator of this substitution group is of order 4 and has for its square any one of the substitutions of order 2 contained in it, since all of these substitutions of order 2 are conjugate under this substitution group. As this substitution of order 4 is also commutative with the given invariant substitutions of order 3 and must involve three cycles of order 4 and three transpositions, it must be one of three substitutions. Assuming that its square is

$$dj \cdot em \cdot fi \cdot gn \cdot hk \cdot lo,$$

these three substitutions are as follows:

$$ap \cdot bq \cdot cr \cdot dfji \cdot mleo \cdot hnkg$$

$$ap \cdot bq \cdot cr \cdot dljo \cdot mneg \cdot hfki$$

$$ap \cdot bq \cdot cr \cdot dnjg \cdot mfei \cdot hlko.$$

It is not difficult to verify that the first two of these substitutions together with the above group of order 60 and degree 15 give rise to substitutions which could not appear in the group under consideration. On the other hand, the last one of these three substitutions together with the group of order 60 actually generates a group of order 1,080 and of degree 18. This group is generated by the following substitutions:

$$s_1 = alk \cdot bnj \cdot cfe \cdot dmh \cdot goi,$$

$$s_2 = apk \cdot bqj \cdot cre \cdot dif \cdot gn h \cdot lmo.$$

The fact that these substitutions satisfy the conditions imposed on s_1 and s_2 results from the following equations:

$$s_1 s_2 = amgl \cdot bhin \cdot cdof \cdot er \cdot jq \cdot kp,$$

$$s_1^2 s_2 = dgfre \cdot honqj \cdot ilpkm.$$

The given substitutions satisfy also the conditions imposed on t_1 and t_2 , since

$$t_1 = ak \cdot bj \cdot ce \cdot dril \cdot fhqg \cdot mpon,$$

$$t_1^2 t_2 = alrk \cdot bnpi \cdot cfql \cdot dg \cdot ho \cdot im,$$

$$t_1^3 t_2 t_1 t_2^2 = alcfbn \cdot dkhemj \cdot gio \cdot pqr.$$

The group generated by s_1 and s_2 contains

$$(s_1 s_2 s_1)^2 (s_1 s_2)^2 s_1^2 s_2 = dhm \cdot eno \cdot fgk \cdot ijl \cdot pqr.$$

As the product of this substitution of order 3 and $s_1^2 s_2$ is of order 2, these two substitutions generate a transitive group of order 60 and of degree 15. It therefore results that s_1 and s_2 generate a transitive group of degree 18 whose order cannot be less than 1,080, and from the given abstract properties of s_1 and s_2 it follows that this order cannot exceed 1,080. It has therefore been proved that the G is actually of order 1,080.

In the given group of order 60 each substitution of order 3 is transformed into itself by its own powers only. Hence in the group of order 1,080 such a substitution is transformed into itself by only 9 substitutions. The Sylow subgroups of order 27 contained in this group must therefore be non-abelian. Hence it results that G does not contain a subgroup which is simply isomorphic with the alternating group of degree 6.

It may be noted that to every operator in the central quotient group there corresponds at least one operator of G whose order is equal to the order of the operator of this quotient group, and yet G does not involve any subgroup which is simply isomorphic with the central quotient group. In this respect the present group differs from the group of order 120 which has for its central quotient group the icosohedral group but does not contain the latter as a subgroup. In this group of order 120, operators of order 4 correspond to operators of order 2 in the central quotient group. The Sylow subgroups of order 8 contained in G are separately simply isomorphic with the octic group, just as in the simple group of order 360, while the Sylow subgroups of order 27 are separately simply isomorphic with the non-abelian group of this order which involves no operator of order 9. In particular, the following theorem has been established: *There are two and only two groups, besides identity, which are generated by s_1 and s_2 when these two operators satisfy the conditions*

$$s_1^3 = s_2^3 = (s_1 s_2)^4 = (s_1^2 s_2)^5 = 1.$$

One of these is the simple group of order 360 and the other is a group of order 1,080 whose central quotient group is this simple group but which does not involve this simple group as a subgroup.

UNIVERSITY OF ILLINOIS.

NOTE ON RIEMANN SPACES.

BY DR. JAMES W. ALEXANDER.

(Read before the American Mathematical Society February 28, 1920.)

1. It is proposed to establish the theorem that every closed orientable n -dimensional manifold can be represented on an n -dimensional hypersphere as a Riemann space, or generalized Riemann surface. The argument will be carried through explicitly for the case $n = 3$ only, since the extension to higher dimensions is perfectly automatic and requires scarcely more than the modification of a few subscripts.

2. We know that a 3-dimensional manifold can always be built up out of the points and boundary points of a finite number of tetrahedral regions by suitably matching together in pairs the triangular faces of the bounding tetrahedra. Let $A_1, A_2, \dots, A_k$ be the points of the manifold which correspond to the vertices of the tetrahedra. Then it can always be so arranged that the four points A_{i1}, A_{i2}, A_{i3} , and A_{i4} corresponding to the vertices of a tetrahedron are distinct, in which case, the bounded tetrahedral region may be designated by the symbol $|A_{i1}A_{i2}A_{i3}A_{i4}|$, where the ordering of the letters A_{ij} is immaterial.

We shall say that every permutation of the letters in the symbol for a tetrahedral region determines a sense on the region and that two permutations determine the same or opposite senses according as they differ by an even or an odd number of transpositions. A tetrahedral region becomes sensed, or oriented, by association with one of the permutations and is then conveniently designated either by the symbol for the sense-giving permutation or by the symbol for any other permutation taken with a plus or minus sign according as the second permutation differs from the first by an even or an odd number of transpositions. Thus, we have

$$A_1A_2A_3A_4 = -A_2A_3A_4A_1 = A_3A_4A_1A_2 = \dots,$$

but

$$A_1A_2A_3A_4 \neq A_2A_3A_4A_1, \text{ etc.}$$

3. Now, consider two tetrahedral regions $|AA_1A_2A_3|$ and $|A'A_1A_2A_3|$ with a face $A_1A_2A_3$ in common. If senses be

assigned to these regions, the given senses will be said to be consistent at the face $A_1A_2A_3$ if and only if we have one of the following combinations

$$(1) \quad \begin{array}{ll} AA_1A_2A_3 & \text{and} \quad -A'A_1A_2A_3 \\ -AA_1A_2A_3 & \text{and} \quad A'A_1A_2A_3. \end{array} \quad \text{or}$$

The manifold is said to be orientable if senses can be assigned to all the tetrahedral regions in such a way as to be consistent at every face. The property of being or not being orientable is an invariant of the manifold and does not depend on the selection of the set of tetrahedral regions.* From now on, we shall confine the discussion to orientable manifolds and shall assume that the tetrahedral regions have all been consistently sensed.

4. Now suppose that in the space of inversion (with a single point at infinity, and therefore like the hypersphere topologically) we set up a system of axes and select k finite points $P_i(x_i, y_i, z_i)$, $i=1, 2, \dots, k$, one for each of the points A_i , and such that no four are coplanar. Then, any four of the points, such as P_1, P_2, P_3 and P_4 , determine a tetrahedron which subdivides the space into two tetrahedral regions, one of which does not contain the point at infinity. We designate the latter region by any permutation $P_{i1}P_{i2}P_{i3}P_{i4}$ of the points P_1, P_2, P_3, P_4 such that

$$\Delta(P_{i1}P_{i2}P_{i3}P_{i4}) = \begin{vmatrix} x_{i1} & x_{i2} & x_{i3} & x_{i4} \\ y_{i1} & y_{i2} & y_{i3} & y_{i4} \\ z_{i1} & z_{i2} & z_{i3} & z_{i4} \\ 1 & 1 & 1 & 1 \end{vmatrix} > 0,$$

and the one which contains the point at infinity by any permutation $P_{j1}P_{j2}P_{j3}P_{j4}$ such that

$$\Delta(P_{j1}P_{j2}P_{j3}P_{j4}) < 0.$$

With this notation, the desired Riemann space is obtained at once by merely mapping each sensed region $A_{i1}A_{i2}A_{i3}A_{i4}$ of the manifold on the region $P_{i1}P_{i2}P_{i3}P_{i4}$ of the space of inversion, in such a way, of course, that the vertex A_{i1} always goes over into the point P_{i1} , and that the images of the boundaries of various tetrahedral regions join on to one another properly along the images of their faces and edges.

* In the Poincaré terminology, a necessary and sufficient condition that a manifold be orientable is that it have no coefficient of torsion of order n .

The proof that the mapping just defined gives the desired Riemann space rests on the fact that two determinants

$$\Delta(PP_1P_2P_3) \quad \text{and} \quad -\Delta(P'P_1P_2P_3)$$

have the same or opposite signs according as the points P and P' lie on opposite sides of the plane $P_1P_2P_3$ or on the same side. It therefore follows that for two contiguous regions

$$(2) \quad \begin{array}{ll} PP_1P_2P_3 & \text{and} \quad -P'P_1P_2P_3 \quad \text{or} \\ -PP_1P_2P_3 & \text{and} \quad P'P_1P_2P_3 \end{array}$$

the points of one region in the neighborhood of a point of the face $P_1P_2P_3$ are on opposite sides of the face from those of the other region. Hence, by comparison of (1) and (2), it is seen that the mapping is uniform not only in the neighborhood of points of the manifold within a tetrahedral region but also of points on a bounding face. Therefore, the only singularities in the mapping correspond to vertices A_i or edges A_iA_j . Furthermore, the mapping is everywhere r to 1 on the space of inversion, except perhaps at the points P_i or along the edges P_iP_j , because any two points, B and C , not on a line P_iP_j can be joined by a path p which does not meet a line P_iP_j . Therefore, if the point B corresponds to n points $B_1, B_2, \dots, B_n$ of the manifold, the path p corresponds to n non-intersecting paths leading from the B_i 's to an equal number of points C_i .

5. In the 3-dimensional case, a Riemann space obtained by the above construction contains, in general, a network of branch lines at each of which two or more sheets coalesce. It is easy to show that, without modifying the topology of the space, the branch system may be replaced by a set of simple, non-intersecting, closed curves such that only two sheets come together at a curve. The curves may, however, be knotted and linking.

Three-dimensional Riemann spaces have been discussed by Heegaard* and Tietze,† but neither of these mathematicians seems to have been aware of their complete generality.

NEW YORK,
February 22, 1920.

* Heegaard, Inaugural Dissertation, Copenhagen, 1898. Also: *Bulletin Société math. de France*, vol. 44 (1916) p. 161.

† Tietze, *Monatshefte für Mathematik und Physik*, vol. 19 (1908), p. 1. Cf. remark on p. 104.

SHORTER NOTICES.

Junior High School Mathematics. By GEORGE WENTWORTH, DAVID EUGENE SMITH, AND JOSEPH CLIFTON BROWN. Book 3; 282 pages, price 96 cents. Boston, Ginn, 1918.

FOR many years past, the unification of elementary mathematics has been a constant subject of discussion. Until recently, however, very little progress has been made in the direction of a practical solution of this problem. Only in vocational education has there been any definite attempt to modernize and coordinate instruction in mathematics, and in this field results are necessarily of somewhat limited application. Everyone interested in educational affairs must recognize the necessity of keeping pace with the growing needs and opportunities of our time. The members of the American Mathematical Society as well as teachers of high school mathematics should therefore feel indebted to the authors of the work under review for having so effectively modernized high school mathematics, especially geometry, and for the attractive and teachable form in which the results are here presented.

The two books of the series which precede the one under review were intended for the seventh and eighth grades, and include arithmetic, the elements of algebra, and intuitive geometry. Book 3 is based on this preliminary work, and includes algebra, the elements of plane trigonometry, and demonstrative geometry.

In particular, the first section, on algebra, includes the four fundamental processes applied to monomials, polynomials and fractions, the solution of simple and quadratic equations, and of simultaneous linear equations. Algebra is here presented as a universal shorthand, and approached from the practical standpoint of the formula. A pedagogical feature of special importance is the clear and concise treatment, and the use of the unit page, no topic overrunning the page, making each page complete in itself.

The section on plane trigonometry defines the trigonometric functions, and illustrates their practical application to the solution of right triangles. The importance of this section, of course, does not consist in its mathematical content, but

in the stimulus it gives to effort by opening a new and practical field of applications intelligible to every child. As this involves both geometrical and arithmetical magnitudes, it also naturally serves as the connecting link between what precedes on algebra and what follows on geometry.

The concluding section on demonstrative geometry, however, is the one of greatest pedagogical importance, as it makes a radical departure from the canon of Euclid. In the first place, there is no attempt to base the subject on a minimum number of axioms. For instance, it is assumed that at a point on a straight line one perpendicular and only one can be erected. Moreover, angles and triangles are compared and classified, and several problems in construction are explained before there is any attempt at formal demonstration. The axioms are simply the four relating to equals combined with equals, and are all stated on one unit page and illustrated numerically. The postulates are ten in number, and are also listed on one page, and include many statements commonly required to be proved. This variation from Euclid is extremely important, as children frequently get a lasting distaste for, and misconception of, geometry by being required to prove statements which are perfectly obvious, such as that a circle is bisected by a diameter. The book also calls attention to the fact that the axioms here introduced were previously applied in algebra, thereby impressing the child with the important fact that these two subjects have a common mathematical basis.

A very clear explanation is then given of the nature of a proof, and how an inference may be examined and proved. This is an especially meritorious feature, as there are few teachers capable of presenting the true nature of geometrical reasoning in such a clear and concise manner. The nature of an inference is further illustrated by comparison of a geometrical inference with one based on medical diagnosis, or on legal evidence. This is certainly calculated to give training in correct thinking, generally recognized to be so much needed at the present time.

Other features of the book show painstaking care and unusual teaching ability, such as the selection of interesting and practical problem material and the specially prepared set of full page perspective drawings, relating the plane figures of geometry to our three-dimensional world of experience.

The work certainly marks the greatest advance in the pedagogy of elementary mathematics which has yet appeared, and is deserving of a wide adoption as a text.

S. E. SLOCUM.

Business Arithmetic. By C. W. SUTTON and N. J. LENNES. Allyn and Bacon, 1918.

THIS book begins in the most elementary manner with the most elementary operations of arithmetic and covers in great detail their applications to almost every kind of a business transaction which one could imagine which calls for such elementary operations. A printed list of topics which are considered would be far too long to give here. So much information of a business nature is given and given so carefully that the book should prove valuable to any one, particularly a business man, as a book of reference; it is veritably a compendium of business knowledge. However, as a textbook, it violates the well-developed belief that better results are obtained by presenting relatively few fundamental principles and facts and then devoting the rest of the time to stimulating independent thought, rather than by the statement and explanation of a multitude of fairly independent principles and facts. It may be that business arithmetic is an exception. Only those who are familiar with the great number of terms and expressions involved can appreciate the difficulties of attempting to treat the subject in accordance with the theory advanced above.

There is a multitude of problems—there are over one hundred problems on some pages—but the problems are so elementary and frequently so similar that it seems at times that sufficient drill could be obtained by a smaller number and valuable time thus gained.

It is the reviewer's belief that any textbook on elementary mathematics—and especially one on business arithmetic—should include at least a simple statement or discussion of some of the rules for numerical computation. It is with considerable misgivings that we note (in Art. 147) the instructions given the student in adding the numbers 12., 1.49, .978, 640.2, 4.904, .007, which will surely serve to authorize him to ignore the adopted convention followed in writing numbers to indicate the degree of the accuracy. Twelve problems follow on page 115 establishing firmly this dangerous fault which is already all too prevalent.

The book is carefully written and printed and is excellent for purposes of a thorough drill in business principles, but it is not conducive to rigorous and independent thinking; It would probably prove very useful as a textbook in a business school.

C. H. FORSYTH.

Empirical Formulas. By THEODORE R. RUNNING. No. 19 of Mathematical Monographs edited by Mansfield Merriman and Robert S. Woodward. New York, Wiley, 1917. 144 pp.

ONE of the problems of the engineer, chemist, physicist or statistician is the finding of a formula, simple as possible, by means of which an approximate value of one variable may be computed from a given value of another variable. The volume under review is concerned with this indeterminate problem, which is not necessarily the problem of determining the physical law connecting the variables in question. In the first five chapters, which make up the principal part of the book, twenty different forms of equations are taken up, tests are given to show when each type of equation may be suitable for given data, and methods for determining constants are discussed and in many cases illustrated by a concrete example worked out in detail.

For example, one form of equation discussed, No. XVI, is

$$(x + a)(y + b) = c.$$

To test a given set of data thought to follow this law, we plot

$$x - x_k, \quad \frac{x - x_k}{y - y_k},$$

where x_k, y_k are any two corresponding values from the data. If these points lie on a straight line, then it may be assumed that x and y are related as above and we may proceed to find the constants. To illustrate this case the results of a set of experiments giving the potential difference and the current in an electric arc are used and the problem worked out in detail.

Included in these twenty types of equations is the Fourier's series with a limited number of terms to which a whole chapter is devoted. Twenty figures in an appendix show some of the forms of curves belonging to each equation considered. The

last three chapters are on the method of least squares, interpolation and numerical integration.

The book is clear and very easy reading even for the undergraduate. There is perhaps too much detail. The presentation on the page is pleasing and there are few errors. The few found when the book was used in a class were not important. The reviewer regrets that the author stopped short of a discussion of the method of moments and of frequency curves. These however could be included in a companion volume.

A. R. CRATHORNE.

NOTES.

THE American association for the advancement of science will hold its seventy-first meeting at Chicago from December 27, 1920, to January 1, 1921. What has been heretofore Section A, mathematics and astronomy, of the Association has been divided into two sections, A, mathematics, and D, astronomy. Of Section A, Professor D. R. CURTISS has been elected vice-president and Professor W. H. ROEVER secretary, Professors DUNHAM JACKSON, A. D. PITCHER, G. A. BLISS, J. M. PAGE, and H. L. RIETZ members of the sectional committee, Professor G. A. MILLER member of the council, and Professor E. V. HUNTINGTON member of the general committee. Professor JOEL STEBBINS has been elected vice-president and Professor F. R. MOULTON secretary of Section D.

THE British association for the advancement of science will meet at Cardiff, beginning August 24, under the presidency of Professor W. A. HERDMAN. Professor A. S. EDDINGTON has been appointed president of Section A, mathematics and physics.

THE Mathematical society of Greece announces the publication of an official journal, the *Bulletin of the Mathematical Society of Greece*, of which the first volume appeared in 1919. Articles for insertion in the *Bulletin* should be addressed to Professor NILOS SAKELLARIOU, Rue Asklipiou 96, Athens.

The following university courses in mathematics are announced:

COLUMBIA UNIVERSITY (summer session, July 6 to August 13).—By Professor EDWARD KASNER: Fundamental concepts of modern mathematics, five hours; Geometric transformations and groups, five hours.—By Professor W. B. FITE: Differential equations, five hours.—By Dr. G. A. PFEIFFER: Introduction to higher algebra, five hours.

COLUMBIA UNIVERSITY (academic year 1920-1921).—By Professor T. S. FISKE: Theory of functions, four hours.—By Professor F. N. COLE: Algebra, four hours.—By Professor D. E. SMITH: History of mathematics, two hours; Practicum in the history of mathematics, four hours.—By Professor EDWARD KASNER: Seminar in differential geometry, two hours.—By Professor W. B. FITE: Differential equations, three hours.—By Dr. J. F. RITT: Selected topics in the theory of functions of a complex variable, three hours.—By Dr. G. A. PFEIFFER: Analysis situs and applications, three hours.

PROFESSOR M. BORN, of the University of Berlin, has been appointed professor of theoretical physics at the University of Frankfurt, as successor to Professor M. VON LAUE, who has been appointed to the same position at the University of Berlin.

PROFESSOR M. KRAUSE has been elected rector of the Dresden technical school.

DR. E. TREFFTZ, of the technical school at Aachen, has been promoted to a full professorship of mathematics.

DR. A. VON BRUNN has been promoted to a professorship at the Danzig technical school.

DR. H. FALCKENBERG, of the technical school at Braunschweig, has been admitted as privat docent at the University of Königsberg.

PROFESSOR W. OLBRICH has been appointed associate professor of descriptive geometry at the Vienna agricultural school.

ASSOCIATE professor A. TAUBER, of the University of Vienna, has been promoted to a full professorship of mathematics.

ACTUARY ARNFINN PALMSTRÖM has been appointed professor of insurance at the University of Christiania.

PROFESSOR L. BIANCHI, of the University of Pisa, and Sir JOSEPH LARMOR, of Cambridge University, have been elected corresponding members of the Paris academy of sciences, in the section of geometry.

MR. S. POLLARD, of Trinity College, has been awarded a Smith's prize for 1920 for his paper on "The Stieltjes integral and its generalizations."

PROFESSOR J. HADAMARD delivered a course of lectures at Yale University, beginning April 30, on "Some topics in linear partial differential equations."

AT Brown University, associate professor H. P. MANNING has retired from active service, after twenty-nine consecutive years as teacher in the department of mathematics. Dr. R. F. BORDEN, of the University of Illinois, has been appointed instructor in mathematics.

PROFESSOR G. D. BIRKHOFF, of Harvard University, has been elected a member of the Danish academy of sciences.

PROFESSOR O. D. KELLOGG has resigned his position in the University of Missouri and has become permanently connected with Harvard University as associate professor of mathematics.

PROFESSOR J. E. ROWE, of Pennsylvania State College, has resigned to accept an appointment as mathematics and dynamics expert in the ordnance bureau of the war department. He is serving as chief ballistician at the Aberdeen Proving Ground.

PROFESSOR W. L. G. WILLIAMS, of William and Mary College, has been appointed instructor in mathematics at Cornell University.

PROFESSOR CHESTER SNOW, of the department of mathematics at the University of Idaho, has resigned to accept a position as physicist in the bureau of standards at Washington.

PROFESSOR A. D. BUTTERFIELD, of the department of mathematics at the Worcester Polytechnic Institute, has resigned to become educational director for the Norton Company.

PROFESSOR WALDEMAR VOIGT, of the University of Göttingen, died December 13, 1919, at the age of sixty-nine years.

PROFESSOR MORITZ CANTOR, of the University of Heidelberg, died April 10, 1920, at the age of ninety-one years.

PROFESSOR G. E. FISHER, of the University of Pennsylvania, died March 28, 1920, at the age of fifty-seven years, after thirty-one years service in the department of mathematics. Professor Fisher became a member of the American Mathematical Society in 1891.

CORRECTION. On page 331 of the April BULLETIN the date of the opening of the first term of the University of Chicago summer session should be June 21.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOREL (E.). See VALLÉE POUSSIN (C. DE LA).

BREUER (S.). Ueber die irreduktiblen auflösbaren trinomischen Gleichungen fünften Grades. (Diss.) Frankfurt a. M., 1918.

BROUWER (L. E. J.). Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. 2ter Teil: Theorie der Punktmengen. Amsterdam, Johannes Müller, 1919. M. 1.60

BÜRKLEN (O.). Aufgaben zur analytischen Geometrie des Raumes. 2te Auflage. Berlin, 1918. M. 1.25

DECKERT (A.). Analytische Geometrie. Breslau, 1918. M. 7.50

DURELL (C. V.). Modern geometry: The straight line and circle. London, Macmillan, 1920. 10 + 145 pp. 6s.

FELIX KLEIN zur Feier seines siebenzigsten Geburtstages. Sonderheft: Die Naturwissenschaften. 25. April 1919. Berlin, Springer, 1919.

HAMMERSTEIN (A.). Zwei Beiträge zur Zahlentheorie. (Diss.) Göttingen, 1919.

- HATTON (J. L. S.). The theory of the imaginary in geometry together with the trigonometry of the imaginary. Cambridge, University Press, 1920. 8vo. 8 + 216 pp. 18s.
- HELM (G.). Die Grundlehren der höheren Mathematik. Zum Gebrauch bei Anwendungen und Wiederholungen zusammengestellt. Neue, verbesserte Ausgabe. Leipzig, 1919. M. 20.40
- HENKLER (P.). Einführung in mathematisches Denken. Berlin, Union, 1919. 55 pp. M. 2.40
- HÖLDER (O.). Die Mathematik im Verhältnis zu den anderen Wissenschaften. Leipzig, 1918.
- JUNKER (F.). Höhere Analysis. 2ter Teil: Integralrechnung. 3te, verbesserte Auflage. Berlin, 1919. M. 1.25
- . Repetitorium und Aufgabensammlung zur Integralrechnung. 3te, verbesserte Auflage. Berlin, 1919. M. 1.25
- KLEIN (F.). See FELIX KLEIN.
- KOENIGSBERGER (L.). Mein Leben. Heidelberg, Carl Winter, 1919. M. 6.00
- KOWALEWSKI (G.). Grundzüge der Differential- und Integralrechnung. 2te, verbesserte Auflage. Leipzig, 1919. 416 pp. M. 12.00
- LIEZMANN (W.). Methodik des mathematischen Unterrichts. 1ter Teil: Organization, allgemeine Methode und Technik des Unterrichts. Leipzig, Quelle und Meyer, 1919.
- NEDER (L.). Zur Konvergenz der trigonometrischen Reihen, einschliesslich der Potenzreihen auf dem Konvergenzkreise. (Diss.) Göttingen, 1919.
- NEPPI (M. A.). Sul luogo geometrico dei punti del piano di un triangolo aventi i triangoli podari ortogonali equivalenti fra loro; involuppi della retta di Simson ottenuta con qualunque angolo. Alessandria, Gazzotti, 1919. 8vo. 12 pp.
- SIMON (M.). Analytische Geometrie des Raumes. 3te Auflage. Neudruck. Berlin, 1919. M. 1.25
- SPIEWECK (B.). Genauere Untersuchung der Kurven $A\kappa + B\tau = C\tau/\kappa$. (Diss.) Halle, 1919.
- VALLÉE POUSSIN (C. DE LA). Leçons sur l'approximation des fonctions d'une variable réelle professées à la Sorbonne. (Collection de Monographies sur la Théorie des Fonctions publiée sous la direction de M. Émile Borel.) Paris, Gauthier-Villars, 1919. 8vo. 6 + 150 pp. Fr. 12.00
- WIDMER (A.). Ueber die Anzahl der Lösungen gewisser Kongruenzen nach einem Primzahlmodul. (Diss., Technische Hochschule, Zürich.) 1919.

II. ELEMENTARY MATHEMATICS.

- AMICI (N.) e MEDICI (S.). Lezioni di aritmetica ed algebra. Volume 2, per il secondo corso dell'istituto tecnico. 2a edizione. Rocca S. Casciano, L. Cappelli, 1919. 8vo. 268 pp. L. 6.00
- ARNOLD (E. E.). See DURELL (F.).
- COMSTOCK (C. E.). See SYKES (M.).

- DURELL (F.) and ARNOLD (E. E.). A first book in algebra. Chicago, Merrill, 1919. 12mo. 5 + 325 + 39 pp.
- GUGLE (M.). Modern junior mathematics. New York, Gregg Publishing Company, 1920. 12mo. Book 1, 9 + 222 pp.; book 2, 11 + 239 pp.
- MARTINI (Z. A.). Trattato di algebra elementare, ad uso degli istituti tecnici. 8a edizione, riveduta. Livorno, R. Giusti, 1920. 8vo. 15 + 379 pp. L. 6.50
- MEDICI (S.). See AMICI (N.).
- PRÉ (L.). Méthode rapide pour le calcul des intérêts et des déductions. Paris, Gauthier-Villars, 1920. 8vo. 134 pp. Fr. 7.50
- ROCQUES DESVALLÉES (H.). Tables logarithmiques et trigonométriques à quatre décimales (argument en arc et en temps) et tables à trois décimales, à l'usage des physiciens et des navigateurs. Paris, Gauthier-Villars, 1920. 8vo. 24 + 72 pp. Fr. 6.00
- SYKES (M.) and COMSTOCK (C. E.). Plane geometry. Chicago, Rand McNally, 1918. 12mo. 12 + 322 pp.

III. APPLIED MATHEMATICS.

- ARDUNIO (E.). Statistica. 5a edizione, riveduta ed ampliata. (Biblioteca degli Studenti.) Livorno, R. Giusti, 1920. 16mo. 8 + 118 pp. L. 1.80
- BARLOW (G.). See POYNTING (J. H.).
- BEAUVAIS (P.). See NÈGRE (F.).
- BRAUER (P.). Ionentheorie. (Mathematisch-physikalische Bibliothek, Band 38.) Leipzig, Teubner, 1919. M. 1.00
- CAMPBELL (N. R.). La théorie électrique moderne. Théorie électronique. Traduit sur la 2e édition anglaise par A. Corvisy. Paris, Hermann, 1919. 464 pp. Fr. 18.00
- CAPART (G.). See WIEWEGER (H.).
- CORVISY (A.). See CAMPBELL (N. R.).
- CRANZ (H.). See HUGERSHOFF (R.).
- EASTWOOD (G. S.) and FIELDEN (J. R.). An algebra for engineering students. London, Arnold, 1920. 7 + 199 + 15 pp. 7s. 6d.
- FIELDEN (J. R.). See EASTWOOD (G. S.).
- GRÜBLER (M.). Lehrbuch der technischen Mechanik. 1ter Band: Bewegungslehre. Berlin, 1919. M. 8.00
- HAUBER (W.). Die Grundlehren der Statik starrer Körper. 5ter Neudruck. Berlin, 1919. M. 1.25
- HAUSSNER (R.). Darstellende Geometrie. 2ter Teil: Perspektive ebener Gebilde; Kegelschnitte. 2te Auflage. Berlin, 1919. M. 1.25
- HENDERSON (R.) and SHEPPARD (H. N.). Graduation of mortality and other tables. (Actuarial Studies, No. 4.) New York, Actuarial Society of America, 1919. 6 + 82 pp. \$1.25

- HUGERSHOFF (R.) und CRANZ (H.). Grundlagen der Photogrammetrie aus Luftfahrzeugen. Stuttgart, 1919. M. 13.20
- LORENZ (H.). Einführung in die Technik. Leipzig, Teubner, 1919. 94 pp. M. 2.15
- LOVE (A. E. H.). A treatise on the mathematical theory of elasticity. Cambridge, University Press, 1920. Royal 8vo. 18 + 624 pp. 37s. 6d.
- LOVETT (W. J.). Applied naval architecture. London, Longmans, 1920. 8vo.
- MAACK (F.). Raumschach. Einführung in die Spielpraxis. Hamburg, F. Maack, 1919.
- MACKENZIE (I. S. F.). A night raid into space: the story of the heavens told in simple words. London, H. Hardingham, 1920. 143 pp. 2s. 6d.
- MATTHEWS (R. B.). The aviation pocket-book for 1919-20. A compendium of modern practice and a collection of useful notes, formulæ, rules, tables, and data relating to aeronautics. 7th edition, revised and enlarged. London, C. Lockwood and Son, 1919. 24 + 536 pp. 12s. 6d.
- MERIZZI (C.). Lezioni di geometria descrittiva. 2a edizione, corretta. (Biblioteca degli Studenti.) Livorno, R. Giusti, 1920. 16mo. 7 + 202 pp. L. 2.70
- MIE (G.). Das Wesen der Materie. I: Moleküle und Atome. 4te Auflage. (Aus Natur und Geisteswelt, Nr. 58.) Leipzig, Teubner, 1919. 122 pp.
- MÖLLER (J.). Nautik. 2te Auflage. (Aus Natur und Geisteswelt, Nr. 255.) Leipzig, Teubner, 1919. 116 pp.
- MOSS (H.). See WATSON (W.).
- MOUNIER (J.). Les graphiques du patron donnant une solution immédiate approchée de tous problèmes de banque, intérêts, escompte, prix de vente, placements. Paris, Gauthier-Villars, 1920. 4to. 24 pp. Fr. 9.00
- MOUREU (C.). See THOMSON (J. J.).
- NÈGRE (F.) et BEAUVAIS (P.). Calcul, construction et essais d'une dynamo à courant continu. Paris et Liège, 1914 (paru en 1919). 8vo. 385 pp. Fr. 22.50
- PLANCK (M.). Einführung in die Mechanik deformierbarer Körper. Zum Gebrauch bei Vorträgen sowie Selbstunterricht. Leipzig, 1919. M. 9.50
- POYNTING (J. H.). Collected scientific papers by J. H. Poynting. Edited by G. A. Shakespear and G. Barlow. Cambridge, University Press, 1920. Royal 8vo. 32 + 768 pp. 37s. 6d.
- RUSCH (F.). Beobachtung des Himmels mit einfachen Instrumenten. 2te Auflage. (Mathematisch-physikalische Bibliothek, Band 14.) Leipzig, Teubner, 1919. 51 pp.
- SCHOUTEN (J. A.). Die direkte Analysis zur neueren Relativitätstheorie. Amsterdam, 1919. M. 3.20

- SCHUDEISKY (A.). Geometrisches Zeichnen. (Aus Natur und Geisteswelt, Nr. 568.) Leipzig, Teubner, 1919. 99 pp.
- SCHURITZ (H.). Die Perspektive in der Kunst Dürers. (Diss., Technische Hochschule, Darmstadt.) Frankfurt a. M., H. Keller, 1919.
- SHAKESPEAR (G. A.). See POYNTING (J. H.).
- SHEPPARD (H. N.). See HENDERSON (R.).
- SUMNER (P. H.). The design and stability of stream-line kite balloons, with useful tables, aeronautical and mechanical formulæ. London, C. Lockwood and Son, 1920. 8 + 146 pp. 10s. 6d.
- TANCOCK (E. O.). Elements of descriptive astronomy. 2d edition. Oxford, Clarendon Press, 1920. 158 pp. 3s.
- THOMSON (J. J.). La théorie atomique. Traduit de l'anglais par C. Moureu. Paris, Gauthier-Villars, 1919. 16mo. 6 + 58 pp. Fr. 2.00
- WATSON (W.) and MOSS (H.). A text-book of physics. Including a collection of examples and questions. London, Longmans, 1920. 8vo. 17s. 6d.
- WIEWEGER (H.). Recueil de problèmes avec solutions sur l'électricité et ses applications pratiques. Traduit par G. Capart. 3e édition, nouveau tirage. 1919. 8vo. 16 + 400 pp. Fr. 22.50
- WITTENBAUER (F.). Aufgaben aus der technischen Mechanik. 1ter Band. Berlin, 1919. M. 14.00

Publications of the American Mathematical Society

Transactions of the American Mathematical Society.

Published quarterly. Subscription price for annual volume, \$5.00; to members of the Society, \$3.75.

Mathematical Papers of [the Chicago Congress, 1893.

Price, \$3.00; to members, \$1.50.

Evanston Colloquium Lectures, 1893. By FELIX KLEIN.

Price, 75 cents.

Boston Colloquium Lectures, 1903. By H. S. WHITE, F.

S. WOODS and E. B. VAN VLECK. Price, \$2.00; to members, \$1.50.

Princeton Colloquium Lectures, 1909. By G. A. BLISS

and EDWARD KASNER. Price, \$1.50; to members, \$1.00.

Madison Colloquium Lectures, 1913. By L. E. DICKSON

and W. F. OSGOOD. Price, \$2.00; to members, \$1.50.

Cambridge Colloquium Lectures, 1916. Part I. By G.

C. EVANS. Price, \$1.50; to members, \$1.00.

**Address all orders to AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.**

Whole No. 288

CONTENTS

	PAGE
The February Meeting of the American Mathematical Society. By Professor F. N. COLE - - - -	337
Some Recent Developments in the Calculus of Variations. By Professor G. A. BLISS - - - -	343
Groups Generated by Two Operators of Order Three Whose Product is of Order Four. By Professor G. A. MILLER	361
Note on Riemann Spaces. By Dr. J. W. ALEXANDER -	370
Shorter Notices - - - - -	373
Notes - - - - -	377
New Publications - - - - -	380

Changes of address of members, exchanges, and subscribers should be communicated at once to the Secretary of the American Mathematical Society, 501 West 116th Street, New York.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to The American Mathematical Society, 41 North Queen St., Lancaster, Pa., or 501 West 116th Street, New York.

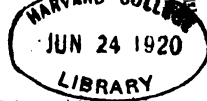
The initiation fees and annual dues of members of the American Mathematical Society are payable to the Treasurer of the American Mathematical Society, Professor J. H. TANNER, Cornell University, Ithaca, N. Y.

Articles for insertion in the BULLETIN should be addressed to the

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.

The following dates have been fixed for the meetings of the Society :

The Twenty-Seventh Summer Meeting and Ninth Colloquium of the Society will be held at the University of Chicago, extending through the week September 6-11, 1920.



BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY

**A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE**

EDITED BY

F. N. COLE	VIRGIL SNYDER	J. W. YOUNG
ALEXANDER ZIWET	D. E. SMITH	
T. LEVI-CIVITA	R. C. ARCHIBALD	

VOLUME XXVI., NUMBER 9

JUNE, 1920

**PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK**

1920

Entered at the Post Office at Lancaster, Pa., as second-class matter
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER

Whole No. 289

Digitized by Google **\$5.00 a year**

Publications of the American Mathematical Society

Transactions of the American Mathematical Society.

Published quarterly. Subscription price for annual volume, \$5.00; to members of the Society, \$3.75.

Mathematical Papers of the Chicago Congress, 1893.

Price, \$3.00; to members, \$1.50.

Evanston Colloquium Lectures, 1893. By FELIX KLEIN.

Price, 75 cents.

Boston Colloquium Lectures, 1903. By H. S. WHITE, F.

S. WOODS and E. B. VAN VLECK. Price, \$2.00; to members, \$1.50.

Princeton Colloquium Lectures, 1909. By G. A. BLISS

and EDWARD KASNER. Price, \$1.50; to members \$1.00.

Madison Colloquium Lectures, 1913. By L. E. DICKSON

and W. F. OSGOOD. Price, \$2.00; to members, \$1.50

Cambridge Colloquium Lectures, 1916. Part I. By G

C. EVANS. Price, \$1.50; to members, \$1.00.

Address all orders to **AMERICAN MATHEMATICAL SOCIETY,**
501 West 116th Street, New York, N. Y.



THE FOURTEENTH WESTERN MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE fourteenth regular western meeting of the American Mathematical Society, being the forty-fifth regular meeting of the Chicago Section, was held at the University of Chicago on Friday and Saturday, April 9 and 10, 1920. The total attendance at this meeting was one of the largest in the history of the Chicago Section, there being about one hundred and fifty persons present at the meeting of Friday afternoon. Among these were the following sixty-nine members of the Society:

Professor G. A. Bliss, Professor H. Blumberg, Professor R. L. Borger, Professor W. C. Brenke, Professor E. W. Brown, Professor W. D. Cairns, Dr. C. C. Camp, Dr. J. W. Campbell, Professor J. A. Caparo, Professor R. D. Carmichael, Professor A. B. Coble, Professor A. R. Crathorne, Mr. H. W. Curjel, Dr. H. B. Curtis, Professor D. R. Curtiss, Professor S. C. Davisson, Dr. W. W. Denton, Professor L. E. Dickson, Professor Arnold Dresden, Professor Arnold Emch, Mr. E. B. Escott, Mr. T. C. Fry, Professor M. E. Graber, Professor Harris Hancock, Professor W. L. Hart, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor G. O. James, Professor A. M. Kenyon, Dr. J. R. Kline, Professor W. C. Krathwohl, Professor J. W. Lasley, Jr., Professor Kurt Laves, Professor A. C. Lunn, Professor J. V. McKelvey, Professor Malcolm McNeill, Professor W. D. MacMillan, Professor Max Mason, Professor T. E. Mason, Professor Bessie I. Miller, Professor W. L. Miser, Professor E. H. Moore, Professor E. E. Moots, Professor E. J. Moulton, Professor F. R. Moulton, Professor G. W. Myers, Professor C. I. Palmer, Professor A. D. Pitcher, Professor S. E. Rasor, Professor P. R. Rider, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Professor D. A. Rothrock, Professor Oscar Schmiedel, Professor G. T. Sellev, Professor E. B. Skinner, Professor H. E. Slaughter, Professor R. B. Stone, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. J. Wilczynski, Professor C. E. Wilder, Professor K. P. Williams, Professor R. E. Wilson, Professor C. H. Yeaton, Professor A. E. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

On Friday evening seventy-four members and guests gathered at a dinner at the Quadrangle Club.

The session of Friday afternoon, presided over by Professors E. W. Brown and D. R. Curtiss, was devoted to a symposium on "The Maxwell field equations and the theory of relativity." Papers were read by Professors Max Mason and A. C. Lunn; the discussion was participated in by Professors E. W. Brown and Alexander Ziwet. Synopses of the symposium papers follow:

I. "Electromagnetic Field Equations," by Professor MAX MASON.

1. Definition of the vectors E , B . The field equations for charge and free space, and the retarded potentials.

2. The Lorentz electron hypothesis. The electromagnetic equations for ponderable bodies; actual and averaged value of the vectors.

3. Comparison with the action at a distance theories (Weber, Clausius et al.) and with the action in the medium theories (Maxwell, Hertz).

4. The equation of activity and of energy. The dynamics of the electron.

5. Types of analytical problems in field determinations.

6. Ponderable matter and free electrons; thermionic space charge and current.

References.

Whittaker, History of the Theories of Aether and Electricity. Encyclopädie der mathematischen Wissenschaften, V. 12 (Reiff and Sommerfeld), V. 13, 14 (Lorentz).

Lorentz, The Theory of Electrons.

II. "The Theory of Relativity," by Professor A. C. LUNN.

1. Terrestrial mechanics and geometric optics on a laboratory scale. The experimental aspect of euclidean metrics, optical meaning of straightness. The materialization of space by the ideal rigid body. Practical chronometers. The group of transformations of newtonian mechanics. The gravitational field. Large scale geodesy, with possible non-euclidean phrasing.

2. Celestial mechanics. Extension of application of the general laws of mechanics and success of the law of gravitation when observations are reduced on the basis of euclidean triangulation and rectilinear path of light. Exception in

the case of the line of apses of Mercury. Determination of inertial systems. The "absoluteness" of rotation.

3. Optics and ether theories. Development of the wave-theory of light in its successive forms. Experimental basis of the notion of fixed ether. Problem of optical determination of absolute motion and negative outcome of the Michelson-Morley experiment. The Fitz-Gerald contraction as a presumptive physical phenomenon.

4. Electromagnetic theory and electrons. Distinct physical effects of ether displacement and polarization currents. The Larmor-Lorentz equations and their group of transformations. Comparison with newtonian relativity.

5. Einstein's first or special form of relativity in terms of the constancy of the velocity of light and the mutual dependence of space and time measurements. Minkowski's theory of euclidean four-dimensional space-time and the corresponding vector analysis or algebraic covariance theory associated with the invariant sum of squares. Real interpretations of non-euclidean type. The tensor of stress, momentum and density.

6. General relativity in terms of covariant theory of general quadratic differential form in four variables. The "local" affine vector analysis of an infinitesimal region associated with the group of all linear transformations. Curvature of finite regions of space-time and application of covariant differentiation to the statement of physical relations. The Riemann-Christoffel four-index tensor and the associated reduced tensor.

7. The general dynamical tensor and Einstein's law of gravitation. Newton's law as a first approximation. Equations of motion for particles in the void. Path of a ray of light. Explanation of the residual in the case of Mercury and prediction of bending of ray of light passing near the sun. Outlook on other phenomena. Relation to molecular theories.

References.

- Einstein: *Annalen der Physik*, volume 17 (1905), pages 891-921 and volume 49 (1916), pages 769-822; *Jahrbuch der Radioaktivität*, volume 4 (1907), pages 411-462.
 Laub and Ishiwara: *Jahrbuch der Radioaktivität*, volume 7 (1910), pages 405-463 and volume 9 (1912), pages 560-648.
 Minkowski: *Göttinger Nachrichten*, 1908, pages 53-111, and *Mathematische Annalen*, volume 68 (1910), pages 472-525.

Eddington: Report on the Relativity Theory of Gravitation. London, Fleetway Press, 1918.

At the sessions of Friday and Saturday forenoons, presided over by Professor Carmichael, chairman of the section, and Professor Wilczynski respectively, the following papers were read:

(1) Professor G. A. MILLER: "Transitive constituents of the group of isomorphisms of an abelian group of order p^m ."

(2) Professor G. E. WAHLIN: "On the unessential divisors of the discriminants of a domain."

(3) Professor E. B. STOFFER: "Seminvariants of a general system of linear homogeneous differential equations."

(4) Professor E. J. WILCZYNSKI: "A set of characteristic properties of a class of congruences connected with the theory of functions."

(5) Professor A. B. COBLE: "Porisms."

(6) Professor O. E. GLENN: "Modular covariants of formal type."

(7) Professor T. E. MASON: "New pairs of amicable numbers."

(8) Professor DUNHAM JACKSON: "On functions of closest approximation."

(9) Professor R. D. CARMICHAEL: "On the expansion of certain analytic functions in series."

(10) Professor R. D. CARMICHAEL: "Boundary value and expansion problems: Algebraic basis of the theory" (preliminary communication).

(11) Professor R. D. CARMICHAEL: "Boundary value and expansion problems: Formulation of various transcendental expansion problems" (preliminary communication).

(12) Professor D. R. CURTISS: "On Jensen's extension of Rolle's theorem."

(13) Professor K. P. WILLIAMS: "The spiral descent of an airplane."

(14) Dr. GEORGE RUTLEDGE: "An invariant area property of polynomial curves of odd degree and the direct evaluation of Cotes's coefficients."

(15) Professor P. R. RIDER: "The minimum area between a curve and its caustic."

(16) Professor W. D. MACMILLAN: "On the moment of inertia in the problem of n bodies."

(17) Dr. J. R. KLINE: "Concerning the boundary of uniformly connected 'im kleinen' domains."

(18) Professor E. H. MOORE: "On the reciprocal of the general algebraic matrix."

(19) Professor W. H. ROEVER: "Determination of the wind structure from the flight of a pilot balloon."

The papers of Professors Miller, Wahlin, Stouffer, Glenn, and of Dr. Rutledge were read by title. Abstracts of the papers, numbered in accordance with the above list, are given below:

1. An abelian group cannot contain more than two characteristic operators. Hence the group of isomorphisms I of an abelian group G , when represented on letters corresponding to non-characteristic operators of G , cannot contain more than two transitive constituents which are separately simply isomorphic with I and a necessary and sufficient condition that it contains two such constituents is that the order of G is twice an odd number. When the order of G is p^m , p being a prime number, then the I of G contains only one transitive constituent which is simply isomorphic with G . Professor Miller investigates the relations existing between the various constituents of I , especially those corresponding to operators of the same order contained in G . In particular, he proves that to the identity of one such constituent there corresponds an abelian group of order p^a and of type $(1, 1, 1, \dots)$ in the next higher constituent, and he determines the orders of these groups.

2. Various investigators have made extensive studies of the unessential divisors of the discriminants of the integers of an algebraic domain. In his *Theorie der algebraischen Zahlen*, Hensel shows that if p is a common unessential divisor of the discriminants of all integers of a domain and if in the domain all the prime factors of p are of the first degree, then p is necessarily less than the degree of the domain. In the case when all the prime factors of p are not of the first degree, he finds a larger upper limit for the primes which may be common unessential divisors. In the *Mathematische Annalen*, volume 73, von Zylinski shows that p is always less than n , if p is a common unessential discriminant divisor. In the present paper, Professor Wahlin shows that in case all the prime divisors of p are of higher degree than the first, a still smaller limit can be fixed.

3. Consider the system of linear homogeneous differential equations

$$y_i^{(m)} + \sum_{l=0}^{m-1} \sum_{k=1}^n \binom{m}{l} p_{ikl} y_k^{(l)} = 0 \quad (i = 1, 2, \dots, n),$$

where p_{ikl} are functions of the independent variable x . It is known that the most general transformation of the dependent variables which leaves this system unchanged in form is given

by the equations $y_k = \sum_{\lambda=1}^n \alpha_{k\lambda}(x) \bar{y}_\lambda$, $|\alpha_{k\lambda}| \neq 0$, where $\alpha_{k\lambda}$ are arbitrary functions of x . A function of the coefficients and their derivatives which has the same value for the original system as for the transformed system is called a seminvariant. Professor Stouffer calculates a complete system of seminvariants for the above set of equations. The seminvariants for the simpler case when $p_{ikm-1} = 0$, were first obtained, and the seminvariants for the general case were obtained from them by simple transformations.

4. If two complex variables, z and w , are transferred to the same sphere, and if the points which correspond to each other in a functional relation between z and w are joined by straight lines, the lines thus obtained form a congruence. The general properties of such congruences were determined by Professor Wilczynski in a paper in the *Transactions*, October, 1919. In the present paper, Professor Wilczynski shows that a certain list selected from these properties is characteristic of such congruences. It is a notable fact that most of these properties are of a purely projective character.

5. Professor Coble's paper is concerned with multiple binary forms which have the closure property. A poristic form of this sort defines a complementary poristic form such that the product of the two can be expressed as a determinant which by its very nature is poristic.

6. Professor Glenn's paper presents in brief the main results developed in the author's memoir on formal invariancy, read before the Society in December 1919 (see abstract in this BULLETIN for March, 1920). Certain further developments of general theory are included.

7. By making use of Lehmer's Table of Primes, Professor Mason has been able to show that the numbers 256·8520191

and 256·257·33023 are a pair of amicable numbers. This pair of numbers belongs to one of the types treated by Euler, but the large prime was beyond the limits of his table of primes. Other number pairs are obtained by Euler's methods or modifications of his methods.

8. The determination of the best polynomial of approximation for a given continuous function $f(x)$ in a given interval, the degree of the polynomial being given, depends on the meaning attached to the phrase "best approximation." The polynomial for which the maximum of the absolute value of the error is as small as possible is known as the Tchebychef polynomial, and has been extensively studied. The polynomial for which the integral of the square of the error is a minimum is the sum of the first terms of the Legendre's series for $f(x)$, and its properties are also well known. Professor Jackson considers the polynomials for which the integral of the m th power of the error is a minimum, where m is any even positive integer, or, more generally, the integral of the m th power of the absolute value of the error, where m is any number greater than 1. It is found that certain theorems which are well known for $m = 2$ are applicable in the more general case; and it is shown that the approximating polynomial corresponding to exponent m approaches the Tchebychef polynomial as a limit when m becomes infinite. The discussion is put in such a form as to apply also to approximation by finite trigonometric sums, for example, and generally to approximation by linear combinations of a given set of linearly independent functions, having such further properties as to insure the uniqueness of the best approximating function in the sense of Tchebychef.

9. For a certain class $S(x)$ of cases of the $\bar{\Omega}$ -series introduced by Professor Carmichael in his paper in the *Transactions*, volume 17, he now treats anew the problem of the representation of functions in such series. The basic function $g(x)$ is restricted to the special asymptotic form

$$g(x) \sim x^{\mu - \rho x} e^{a + \beta x} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots \right),$$

valid in a suitable sector V . The object of the present note is to derive certain necessary and sufficient conditions for the

representation of functions in the form of series $S(x + a)$, where a is a suitable constant depending on the function to be represented.

10. Lying back of several well-known expansion problems in the theory of integral equations and of linear differential equations, both ordinary and partial, is a corresponding problem for linear algebraic equations of which the transcendental problems are limiting cases, each limiting case being realized in a way peculiar to its own class. The object of Professor Carmichael's second paper is to study systematically and to generalize in several directions the algebraic problems which thus come into the foreground, the primary purpose being to come to such understanding of these algebraic matters as will make them serve in the fullest way possible as a guide to the formulation and solution of important transcendental problems of expansion.

11. With the results of his second paper as a guide, Professor Carmichael formulates in his third paper several transcendental expansion problems (in association with certain boundary conditions when these are relevant) having to do with equations and systems of equations of the following classes: linear differential equations, both ordinary and partial; linear difference equations, both ordinary and partial; integral equations; integro-differential equations; integro-difference equations; differentio-difference equations, and some others. The systems mentioned may be made up of equations of only one class or of equations from two or more different classes. This preliminary communication announces the formulation of these problems and a certain small progress toward their solution.

12. The theorem of Jensen according to which the complex roots of the derived functions of a polynomial with real coefficients must lie within ellipses of a certain system gives little further information regarding their location. In the present paper, Professor Curtiss obtains results of a more precise nature, and applies these to the problem of locating the roots of the polynomial when the roots of a derivative are known.

13. The orientation of an airplane when making a spiral descent is determined by two angles. One of these angles is given at once in terms of the angle of attack of the machine,

and the other is obtained from the root of a cubic equation given by Devillers. Professor Williams gives a graphical solution of this cubic equation, and also the explicit expansion of the root in question by means of the formula of Lagrange.

14. Newton, in 1711, and Cotes, in 1722, used polynomial curves for approximating definite integrals. Cotes computed the coefficients which bear his name for polynomials up to degree ten. In the present paper, Dr. Rutledge evaluates, by means of determinants, Cotes's coefficients for the general case of even order. A fact brought out by this evaluation is that the area from $x = a$ to $x = b$ under any polynomial curve of degree $2n + 1$ through $2n + 1$ points equally spaced as to abscissas is an invariant of the set of curves of degree $2n + 1$ through these points.

15. Professor Rider's paper considers the problem of joining two fixed points by a curve which with its caustic and the reflected rays at the fixed points will enclose a minimum area. Euler proposed and solved a similar problem concerning a curve and its evolute. For the evolute problem the minimizing curve is a cycloid. For the caustic problem it is a transcendental curve of more complex type. Its equations are obtained in parametric form. The determination of the arbitrary constants in the solution is considered. Finally, a solution is given of a more general problem which includes the evolute and caustic problems as special cases.

16. For the problem of n bodies, Eddington has given the differential equation for the moment of inertia

$$1/2 \cdot d^2 I / dt^2 = 2T + W,$$

where T is the kinetic energy and $W = -k^2 \sum \sum m_i m_j / r_{ij}$ is the potential energy, so that $T + W = E$ is the total energy. If E is positive it is evident that I'' is always positive and ultimate dispersion is inevitable. In a steady state of motion, $2T + W = 0$, as was remarked by Poincaré. Professor MacMillan supposes that a system of bodies maintains its configuration in undergoing a process of expansion and contraction. Under this hypothesis, W is inversely proportional to the square root of I and the equation for I may be written $1/2 \cdot d^2 I / dt^2 = 2E + k_1^2 / \sqrt{I}$. The integrals of this equation show that if E is negative the period of the oscillation is

$P = k_1^2/(-2E)^{3/2}$, which is independent of the amplitude of the oscillation. If the amplitude of the oscillation be increased to the point where the system collapses ($I = 0$), then the maximum value of $\sqrt{I}$ is twice the value which it has for the steady state in which the amplitude is zero. In other words, the system could not expand to double its size for a steady state without eventually collapsing. Applications are made to star clusters.

17. Dr. Kline shows that if R is a uniformly connected "im kleinen," bounded, two-dimensional domain whose exterior is also uniformly connected "im kleinen," then there exists a finitely connected domain Q whose boundary consists of a finite number of simple closed curves, no two of which have a point in common, such that $Q - R$ is a totally disconnected set. Using this result, he shows that necessary and sufficient conditions that a bounded two-dimensional domain R be finitely connected and bounded by a finite number of simple closed curves, no two of which have a point in common, are (1) that R be fully expanded, and (2) that the exterior of R be uniformly connected "im kleinen." The notion of a set being uniformly connected "im kleinen" on every convergent sequence belonging to the set is then defined. It is shown that any bounded set M , that is uniformly connected "im kleinen" on every convergent sequence belonging to M , is uniformly connected "im kleinen." This is not necessarily true, however, if M is unbounded. Using this notion, it is shown that if R is a proper subset of a two-dimensional euclidean space which is unbounded and simply connected, then a necessary and sufficient condition that R have as its boundary an open curve is that R be uniformly connected "im kleinen" on every convergent sequence.

18. In this paper Professor Moore calls attention to a useful extension of the classical notion of the reciprocal of a non-singular square matrix. Consider any $m \times n$ matrix κ , i.e., an array in m rows and n columns of mn complex numbers $\kappa(uv)$ ($u = 1, \dots, m$; $v = 1, \dots, n$). There exists one and only one $n \times m$ matrix λ , the reciprocal of κ , such that (1) the columns of λ are linear combinations of the conjugates of the rows of κ , (2) the rows of λ are linear combinations of the conjugates of the columns of κ , (3) the matrix $TS\kappa\lambda\kappa$ obtained by matricial composition of the matrices κ, λ, κ is the original

matrix κ : $TS\kappa\lambda\kappa = \kappa$, viz., $\kappa(uv) = \sum_u \kappa(ut)\lambda(ts)\kappa(sv)$ (uv), where u, s run from 1 to m , and v, t run from 1 to n while S ; T denote summations over the respective ranges 1, $\dots$, m ; 1, $\dots$, n . If κ is of rank r , then λ is given explicitly as follows:

$$(r \geq 2) \lambda(v_1 u_1) = \sum_{\substack{u_2 \leq \dots \leq u_r \\ v_2 \leq \dots \leq v_r}} \kappa(\overset{u_1}{u_2} \dots \overset{u_r}{u_r}) \bar{\kappa}(\overset{u_1}{v_1} \overset{u_2}{v_2} \dots \overset{u_r}{v_r}) \\ \div \sum_{\substack{t_1 \leq \dots \leq t_r \\ t_1 < \dots < t_r}} \kappa(\overset{t_1}{t_1} \dots \overset{t_r}{t_r}) \bar{\kappa}(\overset{t_1}{t_1} \dots \overset{t_r}{t_r}) \quad (u_1 v_1);$$

$$(r = 1) \lambda(vu) = \bar{\kappa}(uv) \div \sum_{st} \kappa(st) \bar{\kappa}(st) \quad (uv);$$

$$(r = 0) \lambda(vu) = 0 \quad (uv);$$

where as usual $\kappa(\overset{g_1}{h_1} \dots \overset{g_k}{h_k})$ denotes the determinant of the k^2 numbers $\kappa(g_i h_j)$ ($i, j = 1, \dots, k$) and $\bar{\kappa}(\overset{g_1}{h_1} \dots \overset{g_k}{h_k})$ denotes the determinant of the conjugate numbers $\kappa(\bar{g}_i \bar{h}_j)$.

The relation between κ and λ is mutual: κ is the reciprocal of λ , viz., (4), (5): the columns (rows) of κ are linear combinations of the conjugates of the rows (columns) of λ ; (6) $ST\lambda\kappa\lambda = \lambda$. The linear combinations of the columns of $\kappa(\lambda)$ are the linear combinations of the conjugates of the rows of $\lambda(\kappa)$ and constitute the m -ary vectors μ (the n -ary vectors ν) of a linear r -space M (N) lying in the complete m -space (n -space) of all m -ary (n -ary) vectors. Let $\bar{M}$ ($\bar{N}$) denote the conjugate space of the conjugate vectors $\bar{\mu}$ ($\bar{\nu}$). Then the matrices $\kappa\lambda$ establish 1-1 linear vector correspondences between the spaces M , $\bar{M}$ and the respective spaces N , $\bar{N}$; $\mu = T\kappa\nu$ is equivalent to $\nu = S\lambda\mu$ and $\bar{\nu} = S\bar{\mu}\kappa$ is equivalent to $\bar{\mu} = T\bar{\nu}\lambda$.

19. Professor Roeber shows how, under certain assumptions, it is possible to determine from a single pilot balloon flight the nature of the vector field which represents the wind structure. For this purpose the wind chart and the differential equations of motion of the balloon are needed. These differential equations are obtained from those of a projectile by substituting for the weight of the projectile the lift of the balloon. The wind chart furnishes the finite equations of motion of the balloon. From these two sets of equations the form of the vector field along the path of the balloon is determined, and, under the given assumptions, this is sufficient to

determine the nature of the vector field representing the wind structure.

ARNOLD DRESDEN,
Secretary of the Chicago Section.

THE APRIL MEETING OF THE SAN FRANCISCO SECTION

THE thirty-fifth regular meeting of the San Francisco Section was held at Stanford University on Saturday, April 10. The chairman, Professor Blichfeldt, presided. The total attendance was twenty-three, including the following thirteen members of the Society:

Professor R. E. Allardice, Professor B. A. Bernstein, Professor H. F. Blichfeldt, Professor Thomas Buck, Professor Florian Cajori, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Dr. F. R. Morris, Professor C. A. Noble, Dr. Pauline Sperry.

The action of the executive committee in making plans for a meeting of the Section at the University of Washington June 17-19, in connection with the Pacific division of the American association for the advancement of science was approved. It was decided that this should be a special meeting of the Section, the regular Fall meeting to take place October 23, 1920, as scheduled.

The following papers were presented and discussed:

(1) Professor H. F. BLICHFELDT: "On the approximate representation of irrational numbers" (preliminary report).

(2) Professor FLORIAN CAJORI: "Note on the history of divergent series."

(3) Professor L. M. HOSKINS: "Note on the Lorentz transformation and the notion of time."

(4) Professor E. T. BELL: "The twelve elliptic functions related to sixteen doubly periodic functions of the second kind."

(5) Professor E. T. BELL: "Certain remarkable sums related to 3, 5, and 7 squares."

(6) Professor E. T. BELL: "Parametric solutions for a fundamental equation in the general theory of relativity."

In the absence of the author, Professor Bell's papers were read by title. Abstracts of the papers follow below.

1. The theory of approximate solutions, in terms of integers, of linear equations with irrational coefficients, has been developed mainly by Kronecker and Minkowski. Thus, the equations $x_i - a_i x_n = 0$ ($i = 1, 2, \dots, n-1$) possess an infinite number of sets of integral solutions with errors $\epsilon_i < \frac{n-1}{n} \cdot \frac{1}{x_n^{1/(n-1)}}$. Kronecker has shown that the equations $x_i - a_i x_n = b_i$ ($i = 1, 2, \dots, n-1$) possess an infinite number of sets of integral solutions, with errors ϵ_i at the outset assigned as small as we please, provided no equation exists of the form $\sum_i k_i a_i = k_n$, where the k_i, k_n are integers. Professor Blichfeldt finds that we cannot in general assume $\epsilon_i < N x_n^{-1/(n-1)}$, where N is any given constant.

2. Professor Cajori pointed out that, as early as 1843, A. De Morgan advanced a summation formula for divergent series, discussed asymptotic series and used a principle relating to divergent series; these researches entitle him to the rank of a pioneer in the development of the modern theory of divergent series.

3. Professor Hoskins called attention to an apparent paradox resulting from Einstein's interpretation of the Lorentz transformation. If two exactly similar clocks C and C' keep time correctly for a system S while both are at rest at a point A of the system, and if C remains at A while C' is carried in any closed path with uniform speed v and brought back to A , then according to Einstein the elapsed time as indicated by C' will be less than that indicated by C , the relation being approximately $T' = T[1 - \frac{1}{2}(v/V)^2]$. It is, however, equally legitimate to refer the motion to a coordinate system with respect to which C' remains stationary while C describes a closed path with uniform speed v , and from this point of view the reasoning of Einstein leads to the conclusion that $T = T'[1 - \frac{1}{2}(v/V)^2]$. There seems no ground for regarding the reasoning as valid in one case and not in the other. Is it not possible that in both cases it is based upon an illegitimate physical interpretation of the Lorentz transformation?

4. The twelve elliptic functions considered by Glaisher and other writers are shown in Professor Bell's paper to arise as degenerate forms from a set of sixteen doubly periodic functions of the second kind which may be generated by the operations of a certain group from any one of them. The same functions give also, by differentiation, transformations of the second order, etc., all of the doubly periodic functions of the third kind considered by Hermite, Biehler, Appell, and others, with many more. The paper will appear in the *Messenger of Mathematics*.

5. In connection with his recent determinations of the numbers of representations of integers as sums of 3, 5, 7, 9, 11, and 13 squares, (which will appear in the *American Journal*), Professor Bell remarked that incidentally the 3-square results yield the class number by finite processes only. In this paper new binary class number formulas and analogous results related to quinary and senary forms are derived.

6. The equation solved in Professor Bell's note is that for the square of the line element in the general non-euclidean time-space, which holds a central position in Einstein's theory of gravitation. The solution is not free from quadratures. It is shown that particular solutions of certain other equations in general dynamics may be similarly found. The note will appear in the *Philosophical Magazine*.

B. A. BERNSTEIN,
Secretary of the Section.

THE SUM OF THE FACE ANGLES OF CERTAIN POLYHEDRONS IN n -SPACE.

BY DR. HARRIS F. MACNEISH.

(Read before the American Mathematical Society December 30, 1919.)

A SIMPLE polyhedron in n -space is defined as a set of r -spaces $n-iC_{r-i}$ of which intersect in each i -space; $r > i$; $r = 1, 2, \dots, n-1$; $i = 0, 1, \dots, n-2$.

For a simple polyhedron P^n in n -space we denote the sum of the angles of all the plane faces by S_{p^n} , the number of

vertices by p_0^n , the number of line edges by p_1^n , and the number of r -space faces by p_r^n ; $r = 2, 3, \dots, n-1$.

For a simple polyhedron in n -space we develop the following formula for the sum of the face angles:

$$S_{p^n} = \left[\frac{1}{3} n(n-1) p_0^n - \frac{4}{n-2} p_3^n \right] \pi.$$

For space of three dimensions for any simple polyhedron P^3 we have the Euler formula

$$S_{p^3} = (p_0^3 - 2)2\pi.$$

In space of four dimensions we consider any simple polyhedron P^4 . In any 3-space face A_i^3 ; $S_{A_i^3} = (a_{0,i}^3 - 2)2\pi$. Taking the sum for all of the 3-space faces of P^4 , since in a simple polyhedron four 3-space faces pass through each vertex, and two 3-space faces through each plane,

$$2S_{p^4} = (4p_0^4 - 2p_3^4)2\pi, \quad \text{or} \quad S_{p^4} = (2p_0^4 - p_3^4)2\pi.$$

To obtain an induction proof we assume the generalization of this formula for space of r -dimensions, i.e.,

$$S_{p^r} = \left[\frac{1}{3} r(r-1) p_0^r - \frac{4}{r-2} p_3^r \right] \pi.$$

In $(r+1)$ -space the sum of the plane face angles of each r -space face P_i^r of a simple polyhedron P^{r+1} is given by the above formula.

Taking the sum of the plane face angles for all the r -space faces of the polyhedron P^{r+1} , since (a) $r+1$ r -space faces pass through each vertex; (b) $r-2$ r -space faces pass through each 3-space and (c) $r-1$ r -spaces pass through each plane, in a simple polyhedron in $(r+1)$ -space

$$S_{p^{r+1}} = \left[\frac{1}{3} r(r+1) p_0^{r+1} - \frac{4}{r-1} p_3^{r+1} \right] \pi,$$

which completes the induction proof.

ON BORDERED FREDHOLM DETERMINANTS.

BY PROFESSOR T. H. HILDEBRANDT.

IN a paper on projective transformations in function space, L. L. Dines* sets up certain expressions which he calls bordered Fredholm determinants, and minors. These determinants have properties which are analogous to the properties of Fredholm determinants. It is the purpose of the present note to point out that these bordered Fredholm determinants can be regarded as the Fredholm determinants of a system or a mixed system of linear integral equations, and that as a consequence their properties can be deduced from the corresponding properties of Fredholm determinants.

The bordered determinants arise in the question of the inversion of the transformation

$$(1) \quad \psi(x) = \frac{\alpha(x) + \beta(x)\varphi(x) + \int_a^b \gamma(x, y)\varphi(y)dy}{\delta + \int_a^b \epsilon(y)\varphi(y)dy},$$

where δ is a constant, $\alpha, \beta, \epsilon, \varphi$ and ψ are functions in the interval $I: a \leq x \leq b$, which we assume to be continuous, and $\gamma(x, y)$ is continuous in the square $S: a \leq x \leq b, a \leq y \leq b$. It is also assumed that β does not vanish in the interval I ; as a consequence the question of the inversion of the transformation (1) is reducible to the inversion of the special transformation in which β is taken equal to unity on I . The existence of a unique inverse function φ for every ψ is equivalent to the existence of a unique solution of the system of equations

$$(2) \quad \begin{aligned} \psi_1(x) &= \varphi_1(x) + \alpha(x)\varphi_2 + \int \gamma(x, y)\varphi_1(y)dy, \dagger \\ \psi_2 &= \delta\varphi_2 + \int \epsilon(y)\varphi_1(y)dy, \end{aligned}$$

where φ_2 and ψ_2 are constants and $\varphi = \varphi_1/\varphi_2$ and $\psi = \psi_1/\psi_2$. The determination of φ_1 and φ_2 from this system is in turn

* Cf. *Transactions Amer. Math. Society*, vol. 20 (1919), pp. 45-65.

† Here, as in the sequel, we omit the limits of integration (a, b).

equivalent to the question of solving the system of linear integral equations

$$\psi_1 = \varphi_1 + \int \kappa_{11} \varphi_1 + \int \kappa_{12} \varphi_2,$$

$$\psi_2 = \varphi_2 + \int \kappa_{21} \varphi_1 + \int \kappa_{22} \varphi_2,$$

where

$$\kappa_{11}(x, y) = \gamma(x, y), \quad \kappa_{12}(x, y) = \frac{\alpha(x)}{b-a}, \quad \kappa_{21}(x, y) = \epsilon(y)$$

and

$$\kappa_{22}(x, y) = \frac{\delta - 1}{b-a}.$$

On account of the nature of the functions κ_{12} , κ_{21} , and κ_{22} , every determinant of the Fredholm expansion of the system $\begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix}$ which contains more than one row or column involving κ_{12} , κ_{21} , κ_{22} vanishes identically, and the expansion reduces to

$$1 + \sum_n \frac{1}{n!} \int \cdots \int |\gamma(x_i, x_j)| dx_1 \cdots dx_n \\ + \sum_n \frac{1}{n!} \int \cdots \int \begin{vmatrix} \delta - 1 & \alpha(x_i) \\ \epsilon(x_i) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n \\ (i, j = 1, \dots, n),$$

which on combination becomes

$$1 + \sum_n \frac{1}{n!} \int \cdots \int \begin{vmatrix} \delta & \alpha(x_i) \\ \epsilon(x_j) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n \quad (i, j = 1, \dots, n),$$

which is exactly the expression for Dines's bordered determinant B .

B being then in fact a Fredholm determinant, it follows that we can derive the properties of B from those of Fredholm determinants. For instance, it is at once apparent that there is a product theorem between determinants B based on different functions. In particular since the product of the Fredholm determinants of the two systems $\begin{pmatrix} \kappa_{11}^{(1)} & \kappa_{12}^{(1)} \\ \kappa_{21}^{(1)} & \kappa_{22}^{(1)} \end{pmatrix}$ and $\begin{pmatrix} \kappa_{11}^{(2)} & \kappa_{12}^{(2)} \\ \kappa_{21}^{(2)} & \kappa_{22}^{(2)} \end{pmatrix}$ is the system

$$\begin{pmatrix} \kappa_{11}^{(1)} + \kappa_{11}^{(2)} + \int \kappa_{11}^{(1)} \kappa_{11}^{(2)} + \int \kappa_{12}^{(1)} \kappa_{21}^{(2)}, \\ \kappa_{21}^{(1)} + \kappa_{21}^{(2)} + \int \kappa_{21}^{(1)} \kappa_{11}^{(2)} + \int \kappa_{22}^{(1)} \kappa_{21}^{(2)}, \\ \kappa_{12}^{(1)} + \kappa_{12}^{(2)} + \int \kappa_{11}^{(1)} \kappa_{12}^{(2)} + \int \kappa_{12}^{(1)} \kappa_{22}^{(2)} \\ \kappa_{22}^{(1)} + \kappa_{22}^{(2)} + \int \kappa_{21}^{(1)} \kappa_{12}^{(2)} + \int \kappa_{22}^{(1)} \kappa_{22}^{(2)} \end{pmatrix}$$

we can substitute in this the corresponding values for the κ and find that the B -determinant of the functions $\delta_1, \alpha_1, \epsilon_1, \gamma_1$ multiplied by the B -determinant of $\delta_2, \alpha_2, \epsilon_2, \gamma_2$ is the B -determinant of $\delta_1\delta_2 + \int \epsilon_1\alpha_2, \alpha_2 + \int \gamma_1\alpha_2 + \alpha_1\delta_2, \epsilon_1 + \int \epsilon_1\gamma_2 + \delta_1\epsilon_2, \gamma_1 + \gamma_2 + \int \gamma_1\gamma_2 + \alpha_1\epsilon_2$, which is the result obtained by Dines by more cumbersome methods.

In order to complete the inversion problem it is necessary to introduce minors of the first order. There will be four of these in a system of two integral equations. If we set up these determinants for the special kernels under consideration and note as before that every determinant in the expansion which contains more than one row or column of the α, ϵ and δ is identically zero, we easily obtain the following expressions:

$$F_{11} = - \begin{vmatrix} \delta & \alpha(x) \\ \epsilon(y) & \gamma(x, y) \end{vmatrix} - \Sigma_n \frac{1}{n!} \int \cdots \int \begin{vmatrix} \delta & \alpha(x) & \alpha(x_i) \\ \epsilon(y) & \gamma(x, y) & \gamma(x_i, y) \\ \epsilon(x_j) & \gamma(x, x_j) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n$$

($i, j = 1, \dots, n$),

$$F_{12} = - \frac{1}{b-a} \left[\alpha(x) + \Sigma_n \frac{1}{n!} \int \cdots \int \begin{vmatrix} \alpha(x) & \alpha(x_i) \\ \gamma(x, x_j) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n \right]$$

($i, j = 1, \dots, n$),

$$F_{21} = - \epsilon(y) - \Sigma_n \frac{1}{n!} \int \cdots \int \begin{vmatrix} \epsilon(y) & \gamma(x_i, y) \\ \epsilon(x_j) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n$$

($i, j = 1, \dots, n$),

$$F_{22} = - \frac{1}{b-a} \left[\delta - 1 + \Sigma_n \frac{1}{n!} \int \cdots \int \begin{vmatrix} \delta - 1 & \alpha(x_i) \\ \epsilon(x_j) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n \right]$$

($i, j = 1, \dots, n$).

By comparing with the results of Dines we see that

$$F_{11} = B_1, \quad F_{12} = A/(b - a), \quad F_{21} = E, \\ F_{22} = (D - B)/(b - a).$$

If now we write down the eight reciprocal relations governing the Fredholm determinant and its first minors for a system of equations, as for instance

$$F_{0\kappa_{11}} + F_{11} + \int F_{11\kappa_{11}} + \int F_{12\kappa_{21}} = 0$$

and seven others of a similar nature, and substitute therein the values for our special case, we get at once the reciprocal relations relative to B_1 , A , E , B and D which Dines obtains. It is thus possible to make the discussion of the inversion of the projective transformation on the basis of the Fredholm determinants of a system of linear integral equations.

A more elegant approach to the same results is possible if we note that the system of equations (2) is a mixed system to which the results obtained by Moore* on general linear integral equations apply. Moore discusses not only the case when the functions which enter the equations have the same ranges but also when the ranges are different. Thus in the system under consideration, we really have two distinct ranges, the interval $I: a \leq x \leq b$, with the continuous functions φ_1 , and ψ_1 , and the operation of integration; and the range consisting of a single element with the functions φ_2 , ψ_2 , i.e., a single value, and the operation, the identity operation. By applying the methods sketched by Moore and reducing the results, we obtain at once as the Fredholm determinant of this system, the bordered determinant B , and as first minors, the expressions for B_1 , A , E , and $D - B$. By substituting in the proper reciprocal relations, we arrive at the reciprocal relations obtained by Dines. This method has the advantage of not introducing any extraneous material, i.e., yielding at once the particular quantities which we desire.

This method is also useful in indicating a situation in which one obtains as Fredholm determinants a determinant bordered by n rows and columns. For if we replace the second range which consists of a single element, by a range which consists

* This BULLETIN, vol. 18 (1912), pp. 356 ff.

of a finite number m of elements, we obtain the equations

$$(3) \quad \psi(x) = \varphi(x) + \int \kappa(x, y) \varphi(y) dy + \sum_1^m \kappa_h(x) \varphi_h,$$

$$\psi_g = \varphi_g + \int \lambda_g(y) \varphi(y) dy + \sum_1^m k_{gh} \varphi_h,$$

where φ and ψ are defined on $a \leq x \leq b$, and φ_g and ψ_g on $(1, \dots, m)$. When we set up the Fredholm determinant of this system, we find that all the determinants involving more than m rows or columns of the kernels κ_h , λ_g and k_{gh} vanish identically, and by collecting the non-vanishing elements properly we obtain without much trouble as the Fredholm determinant of this system the expansion

$$|\delta_{gh} + k_{gh}| + \sum_n \frac{1}{n!} \int \dots \int \begin{vmatrix} \delta_{gh} + k_{gh} & \kappa_g(x_i) \\ \lambda_h(x_j) & \kappa(x_i, x_j) \end{vmatrix} dx_1 \dots dx_n$$

$$\left(\begin{matrix} g, h = 1, \dots, m \\ i, j = 1, \dots, n \end{matrix} \right),$$

where $\delta_{gh} = 0$ if $g \neq h$ and unity if $g = h$. Obviously the first and higher ordered minors and the relation between these can be written out without much difficulty.

Systems of linear equations of this last type arise in a number of connections. For instance the mixed linear integral equation

$$\psi(x) = \varphi(x) + \int \kappa(x, y) \varphi(y) dy + \sum_1^m \kappa_g(x) \varphi(x_g),$$

where $x_h, h=1, \dots, m$, are m points on the interval $a \leq x \leq b$, can be reduced to such a system. We need only adjoin the m equations which we obtain by putting $x = x_h$ in this equation and set $\varphi(x_g) = \varphi_g$ in order to obtain such a system. It would follow then that the Fredholm determinant of this mixed integral equation can be written in the form

$$|\delta_{gh} + \kappa_g(x_h)|$$

$$+ \sum_n \frac{1}{n!} \int \dots \int \begin{vmatrix} \delta_{gh} + \kappa_g(x_h) & \kappa_g(x_i) \\ \kappa(x_h, x_j) & \kappa(x_i, x_j) \end{vmatrix} dx_1 \dots dx_n$$

$$\left(\begin{matrix} g, h = 1, \dots, m \\ i, j = 1, \dots, n \end{matrix} \right)$$

with $\delta_{gh} = 0$ if $g \neq h$ and $\delta_{gg} = 1$.

Another situation in which we get such a bordered Fred-

holm determinant is in the system of linear integral equations

$$(4) \quad \begin{aligned} \psi_1 &= \varphi_1 + \int \kappa_{11} \varphi_1 + \int \kappa_{12} \varphi_2, \\ \psi_2 &= \varphi_2 + \int \kappa_{21} \varphi_1 + \int \kappa_{22} \varphi_2, \end{aligned}$$

in which the kernels κ_{12} , and κ_{22} are expressed in terms of the $3n$ functions $\alpha_i(x)$, $\beta_i(x)$, and $\gamma_i(x)$ as follows:

$$\kappa_{12} = \sum_1^n \alpha_i(x) \beta_i(y), \quad \kappa_{22} = \sum_1^n \gamma_i(x) \beta_i(y).$$

If we substitute these values and multiply the second of the two equations by $\beta_j(x)$ and integrate with respect to x , we replace the above system by a system of the type (3) in which $\varphi_j = \int \beta_j(x) \varphi_2(x) dx$. The value of the function φ_2 can then be determined from the second of the equations (4). Since the Fredholm determinant of a system (3) is a bordered determinant, the Fredholm determinant of a system of the type of (4) would also be.

Obviously, these results can be extended by introducing the general range in place of the range $I: a \leq x \leq b$ and general classes of functions instead of continuous functions, and a general linear operator in place of integration in accordance with the postulates of Moore's general theory.

ANN ARBOR, MICH.

ON THE COHERENCE OF CERTAIN SYSTEMS IN GENERAL ANALYSIS.

BY PROFESSOR A. D. PITCHER

IN a previous paper* attention was called to the property *coherence*, so named, of systems $(\mathfrak{Q}; \delta)$, where $\mathfrak{Q}$ is an abstract class of elements q and $\delta(q_1 q_2)$ a distance function defined for every pair $q_1 q_2$ of elements of $\mathfrak{Q}$. In the present paper it is shown that certain of the most notable of the systems of E. H. Moore's Introduction to General Analysis† which were devised without reference to coherence, or indeed without

* "On the foundations of the calcul fonctionnel of Fréchet," by A. D. Pitcher and E. W. Chittenden, *Transactions Amer. Math. Society*, vol. 19, No. 1, pp. 66-78.

† Cf. New Haven Mathematical Colloquium, Yale University Press, 1910. We refer to this memoir as I. G. A.

reference to any property of the range of the independent variable, must nevertheless be coherent.

Consider a system $(\mathfrak{P}; \Delta)$, i.e., an abstract class $\mathfrak{P}$ of elements p with a development Δ^* , and the relation $K_{p_1 p_2 m}^\dagger$ to which the development Δ gives rise. Two sequences $\{p_{1n}\} \{p_{2n}\}$ are said to be connected in case for every integer m there is an integer n_m such that $n \geq n_m$ implies $K_{p_{1n} p_{2n} m}$. An element p is said to be a limit of a sequence $\{p_n\}$, in notation $L_n p_n = p$, in case for every m there is an n_m such that $n \geq n_m$ implies $K_{p_n p m}^\dagger$. A system $(\mathfrak{P}; \Delta)$, or a system $(\mathfrak{P}; K)$, is coherent if it is true that whenever two sequences $\{p_{1n}\} \{p_{2n}\}$ are connected and p is a limit of the sequence $\{p_{1n}\}$ then p is also a limit of the sequence $\{p_{2n}\}$.

We wish to consider the systems $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})^\S$ where the class $\mathfrak{M}$ has the properties $D_1 \Delta K_2$ or the properties $L \Delta K_2$.||

THEOREM. *The system $(\mathfrak{P}; \Delta)$ of a system $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$, where the class $\mathfrak{M}$ has the properties $D_1 \Delta K_2$ or the properties $L \Delta K_2$, is a coherent system.*¶

Since $\{p_{1n}\}$ and $\{p_{2n}\}$ are connected there is a sequence $\{\mathfrak{P}^{\hat{m}_n}\}$ of the development Δ such that $L_n \hat{m}_n = \infty$ and such that for $n \geq n_0$ the elements p_{1n} and p_{2n} each belong to the class $\mathfrak{P}^{\hat{m}_n}$. Since p is a limit of the sequence $\{p_{1n}\}$ there is a sequence $\{\mathfrak{P}^{\bar{m}_n}\}$, of the development Δ , such that $L_n \bar{m}_n = \infty$ and such that for n sufficiently large p and p_{1n} each belong to $\mathfrak{P}^{\bar{m}_n}$.

We must show that p is a limit of the sequence $\{p_{2n}\}$. Suppose this is not true. Then there is an integer m' such that for every n_0 there is an $n > n_0$ such that p and p_{2n} are not in the relation $K_{p p_{2n} m'}$. Thus by permitting n_0 to assume the values 1, 2, 3, . . . , k , . . . and choosing each time the smallest n available we secure a sequence $\{n_k\}$ such that for no k is the relation $K_{p p_{2n_k} m'}$ holding. There is then a sequence $\{p_{2n_k}\}$, a subsequence of $\{p_{2n}\}$, such that for no m exceeding

* Cf. I. G. A., § 75.

† I. G. A., § 77.

‡ Cf. "A contribution to the foundations of the calcul fonctionnel of Fréchet," by T. H. Hildebrandt, *Amer. Jour. of Mathematics*, vol. 34 (1912), pp. 237-290.

§ I. G. A., §§ 75-80.

|| For properties D_1, L, Δ, K_2 of classes $\mathfrak{M}$ see I. G. A., §§ 15, 22, 79, 72.

¶ It should be noted that the property D implies the property D_1 and on that account the systems $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$, where $\mathfrak{M}$ has the properties $D \Delta K_2$, are included in the above theorem.

m' and for no k are p_{2n_k} and p in the same subclass $\mathfrak{P}^{m'}$ of the development Δ .

Now consider the element p and those functions of the developmental system $((\delta^{m'}))$ which are associated with the element p . That is, consider the system $((\delta^{m_0}))$ where $(\mathfrak{P}^{m_0})$ is the system of classes of stage m which contain p . Denote by θ^m the sum of the functions δ^{m_0} which are associated with p at stage m . Then, by the first condition* on the developmental system,

$$L_m \theta_p^m = 1.$$

In case $\mathfrak{M}$ has either the property L or the property D_1 the function θ^m has the property $K_2 \mathfrak{M}$ †. Consider a function φ which has the property $K_2 \mathfrak{M}$. There is a μ of $\mathfrak{M}$ such that for every positive number e there is an m_e such that $K_{p_1 p_2 m_e}$ implies $|\varphi_{p_1} - \varphi_{p_2}| \leq e |\mu_{p_1}|$. Since $\{p_{1n}\}$ and $\{p_{2n}\}$ are connected it follows that for every e there is an n_e such that $n \geq n_e$ implies $|\varphi_{p_{1n}} - \varphi_{p_{2n}}| \leq e |\mu_{p_{1n}}|$. Since the sequence $\{p_{1n}\}$ has p for a limit and μ has the property $K_2 \mathfrak{M}$, the sequence $\{\mu_{p_{1n}}\}$ of values is bounded. Thus for every e there is an n_e such that $|\varphi_{p_{1n}} - \varphi_{p_{2n}}| \leq e$. Therefore $L_n \varphi_{p_{1n}} = L_n \varphi_{p_{2n}} = \varphi_p$.

If the conclusion concerning φ be applied to each function θ^m , we have

$$L_n \theta_{p_{1n}}^m = L_n \theta_{p_{2n}}^m = \theta_p^m.$$

Then for every m

$$L_k \theta_{p_{2n_k}}^m = \theta_p^m$$

and

$$L_m (L_k \theta_{p_{2n_k}}^m) = L_m \theta_p^m = 1.$$

According to the condition (1b) on the developmental system there is for every μ a function μ_0 such that for every e there is an m_e such that for $m \geq m_e$ and for every p

$$\Sigma_h |\mu_{r m h} \delta_p^{m h}| \leq e |\mu_{0p}|.$$

Consider an element p_{2n_k} . Recall that $\theta^m = \Sigma_0 \delta^{m_0}$. For m sufficiently large no δ^{m_0} is associated with a class $\mathfrak{P}^{m_1}$ to which p_{2n_k} belongs (since δ^{m_0} is associated with a class $\mathfrak{P}^{m_0}$ to which p belongs). Since $\mathfrak{M}$ has the properties L , Δ , K_2 or the properties D_1 , Δ , K_2 there is a function μ of $\mathfrak{M}$ which, for m sufficiently large and a number a properly chosen, is such

* I. G. A., § 78 (1a).

† I. G. A., § 72 (2).

that $a\mu_{r^{mg}} \geq 1$, where r^{mg} is the representative element of any class $\mathfrak{P}^{mg}$ to which p belongs. Using such a function μ in the condition (1b) on the developmental system and noting that the sequence $\{\mu_{0p_{2n_k}}\}$ is bounded, we see that for every ϵ there is an m_ϵ such that for $m \geq m_\epsilon$ and for every k the value $\theta_{p_{2n_k}}^m$ does not exceed ϵ . But this affords a contradiction to the conclusion reached above that

$$L_m(L_k\theta_{p_{2n_k}}^m) = 1.$$

Therefore the hypothesis that p is not a limit of the sequence $\{p_{2n}\}$ is contrary to fact.

These considerations may be extended to the infinite developments of Chittenden.*

ADELBERT COLLEGE,
WESTERN RESERVE UNIVERSITY.

ON THE ADJOINT OF A CERTAIN MIXED EQUATION.

BY DR. R. F. BORDEN.

(Read before the American Mathematical Society December 30, 1919.)

CONSIDER a function of the form

$$(1) \quad F\{f(x)\} = \Delta f'(x) + a(x)f'(x) + b(x)\Delta f(x) + c(x)f(x),$$

where $a(x)$, $b(x)$, and $c(x)$ are analytic functions of x , also $\Delta f(x) = f(x+1) - f(x)$, and $f'(x) = (d/dx)f(x)$. We will say that $G\{g(x)\}$ is the adjoint of $F\{f(x)\}$ if $G\{g(x)\} = 0$ is the condition that

$$(2) \quad \int \Sigma g(x)F\{f(x)\}dx = \Sigma M(x) + \int N(x)dx,$$

where Σ denotes an inverse of Δ .

This condition (2) is satisfied if

$$(3) \quad g(x)F\{f(x)\} = \frac{d}{dx}M(x)dx + \Delta N(x),$$

* Cf. "Infinite developments and the composition property $(K_{12}B_1)_*$ in general analysis," by E. W. Chittenden, *Rendiconti del Circolo Matematico di Palermo*, vol. 39 (1915), p. 21, §§ 19 and 21.

where $M(x)$ and $N(x)$ are of the form

$$(4) \quad \begin{aligned} M(x) &= A(x)\Delta f(x) + B(x)f(x), \\ N(x) &= C(x)f'(x) + D(x)f(x). \end{aligned}$$

Equating corresponding coefficients in the two members of (3), we have

$$(5) \quad \begin{aligned} g(x) &= A(x) + C(x), \\ a(x)g(x) &= B'(x), \\ b(x)g(x) &= A'(x) + D(x), \\ c(x)g(x) &= D'(x), \\ \Delta C(x) &= \Delta D(x) = 0. \end{aligned}$$

From (5) we have

$$(6) \quad \begin{aligned} \Delta g'(x) - a(x)g'(x) - b(x+1)\Delta g(x) \\ + [c(x) - a'(x) - \Delta b(x)]g(x) = 0, \end{aligned}$$

which we will call the expression adjoint to the expression (1).^{*} The equations formed by equating (1) and (6) to zero, we will call adjoint equations.

The formal mode of obtaining the coefficients of the adjoint gives (1) as the adjoint to (6) only when $\Delta b(x) = 0$. However this difficulty may be avoided, for $b(x)$ may be taken identically zero without loss of generality, since (1) is transformed by the substitution

$$f(x) = e^{-\int_a^x b(x)dx} h(x)$$

into an expression having no term in $\Delta h(x)$. Accordingly in what follows we will take $b(x)$ as zero when discussing the adjoint.

We will now state a few facts about the equation†

$$(7) \quad \Delta f'(x) + a(x)f'(x) + b(x)\Delta f(x) + c(x)f(x) = 0,$$

where $a(x)$ is not identically one.

^{*} Other forms of adjoint expressions may be found to satisfy (3), but among other advantages (6) affords in what follows the closest analogy to the theory of adjoint partial differential equations. See, for instance, Forsyth: *Theory of Differential Equations*, Vol. 6, p. 112.

† First studied by Poisson: *Journal de l'Ecole Polytechnique*, vol. 6. (1806), pp. 127-141. Its properties here listed are developed in my thesis which is to appear in the *Amer. Jour. of Mathematics*.

(i) The equation (7) has under the group of transformations $f(x) = v(x)g(x)$ two fundamental invariants

$$I(x) = \frac{c(x) - [a(x) - 1]b(x) - a'(x)}{a(x) - 1}$$

and

$$J(x) = \frac{c(x) - [a(x) - 1]b(x - 1)}{a(x) - 1}.$$

(ii) When either $I(x) = 0$ or $J(x) = 0$, the equation may be solved in finite form by the standard methods of solving differential and difference equations, the two processes being separable.

(iii) If n successive applications of the transformation

$$(S) \quad f_{s_1}(x) = f(x + 1) + [a(x) - 1]f(x)$$

or of the transformation

$$(T) \quad f_{t_1}(x) = f'(x) + b(x - 1)f(x)$$

result in an equation with a vanishing invariant, and if n is the least integer for which this is true, then the equation is of rank $n + 1$ with respect to the transformation.

(iv) If I_{s_m} and J_{s_m} are the invariants of the m th transformed equation under (S), and if I_{t_m} and J_{t_m} are the invariants of the equation resulting from m applications of I , then

$$J_{s_n}^s(x) = I_{s_{n-1}}(x) \text{ and } I_{t_n}(x) = J_{t_{n-1}}(x).$$

(v) The equation obtained by applying (S) and (T) in the order named, has the same invariants as has the equation obtained by applying first (T) and then (S).

The adjoint to

$$(8) \quad \Delta f'(x) + a(x)f'(x) + c(x)f(x) = 0$$

is

$$(9) \quad \Delta g'(x) - a(x)g'(x) + [c(x) - a'(x)]g(x) = 0,$$

which has the invariants*

$$i(x) = \frac{c(x) - a'(x)}{-a(x) + 1} + \frac{a'(x)}{-a(x) + 1} = \frac{c(x)}{-a(x) + 1} = -J(x)$$

* The discussion holds only when $a(x) \neq 1$.

and

$$j(x) = \frac{c(x) - a'(x)}{-a(x) + 1} = -I(x).$$

Hence the adjoint equation has the same set of invariants except for order as has the original equation. Consequently if one of two adjoint equations of the form (8) can be solved in finite form on account of a vanishing invariant, the same is true of the other.

Denote equation (8) by $F = 0$ and its n th transform by $F_{s_n} = 0$ or by $F_{\tau_n} = 0$ according as (S) or (T) is used. Using (iv) and (v) we can express the invariants of the transformed equations as follows, each pair of invariants being written under the corresponding equation:

$$\begin{array}{ccccccc} F_{\tau_n} = 0, & \dots, & F_{\tau_1} = 0, & F = 0, & F_{s_1} = 0, & \dots, & F_{s_n} = 0, \\ I_{\tau_n}, & \dots, & I_{\tau_1}, & I, & I_{s_1}, & \dots, & I_{s_n}, \\ I_{\tau_{n+1}}, & \dots, & I_{\tau_2}, & I_{\tau_1}, & I, & \dots, & I_{s_{n+1}}. \end{array}$$

Similarly we have for the adjoint equation $G = 0$, the set

$$\begin{array}{ccccccc} G_{\tau_n} = 0, & \dots, & G_{\tau_1} = 0, & G = 0, & G_{s_1} = 0, & \dots, & G_{s_n} = 0, \\ I_{s_{n+1}}, & \dots, & I, & I_{\tau_1}, & I_{\tau_2}, & \dots, & I_{\tau_{n+1}}, \\ I_{s_n}, & \dots, & I_{s_1}, & I, & I_{\tau_1}, & \dots, & I_{\tau_n}, \end{array}$$

which differs from the set for $F = 0$ and its transforms only in order.

From the two sets we see that, if an equation of the form (8) is of finite rank with respect to either (S) or (T), its adjoint is of the same rank with respect to the other transformation. Also if such an equation is of doubly finite rank, its adjoint is of doubly finite rank with the same index.

The equation

$$(10) \quad \Delta f'(x) + c(x)f(x) = 0$$

is self-adjoint. Its invariants are equal and have the value $-c(x)$. Unlike the corresponding partial differential equation, (10) is not a canonical form for equations with equal invariants. A necessary and sufficient set of conditions that (7) shall have equal invariants is

$$a'(x) = \Delta b(x) = 0.$$

Not all equations of that form are reducible to the form (10); for example

$$\Delta f'(x) + 2f'(x) + c(x)f(x) = 0$$

is not. However, equations with equal invariants which are not self-adjoint do not seem to be of great interest.

From the preceding theorem it follows that *if a self-adjoint equation of form (10) is of finite rank with respect to one of the transformations (S) or (T), it is of the same rank with respect to the other.*

Using the formula*

$$I_{s_n}(x) = I(x) + \sum_1^{n-1} [I(x+k) - J(x+k)] \\ - \Delta \frac{d}{dx} \log \left[\prod_{k=0}^{n-1} I(x+k) \prod_{k=0}^{n-2} I_{s_1}(x+k) \cdots I_{s_{n-1}}(x) \right]$$

and noting that for the self-adjoint case

$$I(x) = J(x) = -c(x),$$

we have

$$-c(x) = \Delta \frac{d}{dx} \log \left[\prod_{k=0}^{n-1} (-c(x+k)) \prod_{k=0}^{n-2} I_{s_1}(x+k) \cdots I_{s_{n-1}}(x) \right]$$

as a necessary and sufficient condition that the self-adjoint equation (10) be of rank $n+1$ with respect to each of the transformations (S) and (T).

UNIVERSITY OF ILLINOIS.

THE SECOND VOLUME OF VEBLEN AND YOUNG'S PROJECTIVE GEOMETRY.

Projective Geometry. By OSWALD VEBLEN and J. W. YOUNG. Boston, Ginn and Company; Vol. 2, by Oswald Veblen, 1918. 12 + 511 pages.

In volume I, Veblen and Young were concerned particularly with those theorems of projective geometry which can be proved on the basis of their assumptions *A* of alignment, assumptions *E* of extension, and an assumption *P* of pro-

* Formula (8) in my thesis, l.c.

jectivity. In some cases use was also made of the assumption H_0 that the diagonal points of a complete quadrangle are non-collinear. A space satisfying A and E is called a general projective space. This term does not seem to the reviewer to be altogether appropriate. According to this terminology real projective spaces, complex projective spaces, etc., are all general projective spaces. This seems rather like saying that every special projective space is a general projective space. It would seem in some respects preferable to omit the word general and say simply that every space satisfying A and E is a projective space.*

In volume II various special projective spaces are studied, each special projective space being a space in which, in addition to A and E , certain other postulates are satisfied. One example of a special projective space is a proper projective space, a space satisfying A , E and P . A projective space satisfying the assumption H that if any harmonic sequence exists not every one contains only a finite number of points is called a non-modular projective space. A modular projective space is one in which every harmonic sequence contains only a finite number of points.

Assumptions A , E , P and H_0 hold true in both the ordinary real and the ordinary complex projective spaces of three dimensions. A space satisfying these assumptions is not necessarily continuous in the sense of Dedekind. Indeed it does not necessarily possess even pseudo-archimedean continuity. Veblen and Young's assumptions A , E and H_0 correspond rather closely to the first fourteen of a set of nineteen postulates published by Pieri in 1899.† The following is a rough translation of Pieri's set.

POSTULATE I°. *Projective points form a class.*

POSTULATE II°. *There exists at least one projective point.*

POSTULATE III°. *If there exists one projective point, there exists at least one other one.*

POSTULATES IV° AND V°.‡ *If a , b are projective points ($a \neq b$), the line ab is a class of points.*

* The same objections would not apply to the use of the term general projective geometry to designate the totality of all theorems that can be proved on the basis of A and E . There is only one such body of theorems and therefore only one such geometry, while there are many sorts of spaces satisfying A and E .

† M. Pieri, "I principii della geometria di posizione composti in sistema logico deduttivo," *Memorie della Reale Accademia delle Scienze di Torino*, vol. 48 (1899), pp. 1-62.

‡ Cf. A₁.

POSTULATE VI°. If a and b are distinct projective points, the line ab is contained in the line ba .

POSTULATE VII°. If a and b are distinct projective points, a belongs to the line ab .

POSTULATE VIII°.* If a and b are distinct projective points, the line ab contains at least one point distinct from a and from b .

POSTULATE IX°. If a and b are distinct projective points and c is a point of the line ab distinct from a , then b belongs to the line ac .

POSTULATE X°. Under the same hypothesis ab contains ac .

POSTULATE XI°.† If $a \neq b$, there exists at least one point not on the line ab .

POSTULATE XII°.‡ If a, b, c are non-collinear projective points and a' is a point of bc distinct from b and from c and b' is a point of ac distinct from a and from c , then the line aa' has a point in common with the line bb' .

POSTULATE XIII°.§ If a, b, c are non-collinear points, there exists a point not in the plane abc .

POSTULATE XIV°.|| If $a \neq b$ and c is a point of ab distinct from a and from b , then the fourth harmonic of c with respect to a and b is distinct from c .

POSTULATE XV°. If a, b, c are distinct points of a line r and d is a point of the line r not belonging to the segment¶ (abc) nor coinciding with a or with c , then d belongs to the segment (bca).

POSTULATE XVI°. If a, b, c are distinct collinear points of a line r and a point d belongs to both of the segments (bca) and (cab), then it does not belong to the segment (abc).

POSTULATE XVII°. If a, b, c are distinct collinear points and d is a point distinct from b on the segment (abc) and e is a point of the segment (adc), then the point e is on the segment (abc).

Postulate XVIII° is a form of Dedekind cut postulate as applied to the points of a segment.

POSTULATE XIX°.** If a, b, c, d are distinct non-coplanar

* Cf. Veblen and Young's E_0 .

† Cf. Veblen and Young's E_1 .

‡ Cf. Veblen and Young's A_1 .

§ Cf. E_1 .

|| Cf. Assumption H_0 . Pieri points out that in the presence of Postulates VI°, XII° and XIII°, Postulate XIV° is equivalent to the proposition that the diagonal points of a complete quadrangle are collinear.

¶ The segment (abc) is defined as the set of all points $[x]$ such that, for some pair of points y and z on the line abc which are harmonic conjugates of each other with respect to a and c , x is the fourth harmonic of b with respect to y and z .

** Cf. Assumption E_1' .

points, then for every point e not in any one of the planes abc , abd , acd , bcd there exists at least one point common to the figures ae , bcd .

It seems not far from accurate to say that Veblen and Young's set A, E, H_0 differs from Pieri's set I° -XIV $^\circ$, XIX $^\circ$ chiefly in that Veblen and Young postulate simply the four assumptions A_1, A_2, E_0 and E_1 instead of Pieri's I-X, which latter may be (perhaps rather roughly) considered as corresponding to the result of breaking up (or analyzing) the group of assumptions A_1, A_2, E_0 and E_1 , in a certain fashion, into a larger number of weaker postulates. Every space that satisfies A, E, H_0 satisfies also I° -XIV $^\circ$ and conversely.

Veblen secures a categorical set of postulates for real projective space by defining the notion of order on a net of rationality and the notion of open cuts in such a net and adding, to A and E , assumption H and the following assumptions C and R .

ASSUMPTION C. *If every net of rationality contains an infinity of points, then on one line l in one net R ($H_0H_1H_\infty$) there is associated with every open cut (A, B) with respect to the scale H_0, H_1, H_∞ , a point $P_{(A, B)}$ which is on l and such that the following conditions are satisfied:*

- (1) *If two open cuts (A, B) and (C, D) are distinct, the points $P_{(A, B)}$ and $P_{(C, D)}$ are distinct;*
- (2) *If (A_1, A_2) and (B_1, B_2) are any two cuts and (C_1, C_2) any open cut between two points A and B of R ($H_0H_1H_\infty$) and if T is a projectivity such that*

$$T(H_\infty AB) = H_\infty P_{(A_1, A_2)} P_{(B_1, B_2)},$$

then $T(P_{(C_1, C_2)})$ is a point associated with some cut (D_1, D_2) between (A_1, A_2) and (B_1, B_2) .

ASSUMPTION R. *On at least one line, if there is one there is not more than one chain.*

Pieri secures the same result by defining the notion of a segment and adding postulates XV $^\circ$ -XVIII $^\circ$.

It seems to the reviewer that one might naturally feel, in view of the complicated nature of Veblen's assumption C , that Pieri's XV $^\circ$ -XVIII $^\circ$ constitute a simpler group of assumptions than that constituted by Veblen's H, C and R . Veblen's set has an apparent advantage in that his assumption C holds true for a complex space as well as for a real one, while this is not true of Pieri's postulate XV $^\circ$. But let us consider

a second article by Pieri, published in 1905.* In this article Pieri gives a set of postulates for complex projective geometry of infinitely many dimensions. The first 10 of these correspond closely to the first 14 of his first set except for the fact that (a) instead of XI° and XIII° of set I we have† in set II a postulate to the effect that for every complex space S_n there exists at least one point not in S_n and (b) postulate XIV° is replaced by a postulate to the effect that the diagonal points of a complete quadrangle are non-collinear.‡

In postulate XII and in most of the remaining postulates XII-XXX use is made of a new undefined notion, the notion of a chain. Postulates XII-XIX are (freely translated) as follows:

POSTULATE XII. *If a, b, c are distinct points of a complex line, the chain of the points a, b, c —indicated by the symbol $|abc|$ —is a class of points belonging to that line.*

POSTULATES XIII AND XIV. *If a, b, c are distinct collinear complex points, the chain $|abc|$ is contained in each of the chains $|acb|$ and $|bac|$.*

POSTULATE XV. *Under the same hypothesis a belongs to the chain $|abc|$.*

POSTULATE XVI. *If a, b, c, d are distinct collinear points and a', b', c', d' are distinct collinear points in perspective correspondence with a, b, c, d , then d belongs to $|abc|$ if and only if d' belongs to $|a'b'c'|$.*

POSTULATE XVII. *If a, b, c are three distinct collinear points, the fourth harmonic of c with respect to a and b belongs to the chain $|abc|$.*

POSTULATES XVIII AND XIX. *If a, b, c, d are four distinct collinear points and d belongs to the chain $|abc|$, then c belongs to $|abd|$ and $|abd|$ is contained in $|abc|$.*

If in the definition of "segment (abc) " quoted above for real space the phrase "line abc " is replaced by "chain $|abc|$ " the resulting definition holds good for complex space.

Postulate XX of set II is equivalent to the proposition obtained by substituting "chain $|abc|$ " for the second "line r " in the statement of postulate XV° of set I. Postulates XXI and XXII of set II are respectively equivalent to XVI° and XVII° of set I.

* "Nuovi principii di geometria proiettiva complessa," *Memorie della Reale Accademia delle Scienze di Torino*, vol. 55 (1905), pp. 189-235.

† A natural change to make in passing from a set of postulates for three dimensions to one for infinitely many dimensions.

‡ Cf. Veblen and Young's H_0 .

The set of postulates I-XXII hold true in real as well as in complex space. If, in this set, postulate X is replaced by XI°, XIII° and XIX° of set I, and Veblen's R and a suitable Dedekind cut postulate are added, there results a categorical set of axioms for real projective geometry of three dimensions. To obtain complex geometry, Pieri adds, to I-XXII, seven more postulates (XXIII-XXX). Of these, XXIII is a sort of pseudo-archimedean axiom whose content may be roughly suggested by saying that it postulates that if a, b, c are three distinct points the net of rationality determined by them is everywhere dense on the segment (abc) .

At first sight it might seem that Veblen and Young's set of assumptions for complex projective geometry contains fewer postulates and is correspondingly simpler than Pieri's set. However, leaving out of consideration some of the earlier axioms in which Pieri gives such a detailed analysis of certain simple relations of alignment, etc., the question arises whether if Veblen's set seems at first sight to be simpler it is not largely because he makes use of a postulate, assumption C, which leaves much to be desired from perhaps at least two points of view. There is a curious difference between the way in which assumption C functions for complex space as compared with the way it functions for real space. Suppose S is a definite real projective space (satisfying A, E and H) whose projective structure is determined in the sense that the question whether or not three points in S lie on the same line has a determinate answer as soon as the three points are themselves determined. If C and R are satisfied in S for one association of points with open cuts as indicated in the statement of assumption C then they are not satisfied for any other such association of points with cuts. That is to say if, in S , (a) on one line l in one net $R(H_0H_1H_\infty)$ there is associated with every open cut (A, B) in $R(H_0H_1H_\infty)$ a point P on l such that conditions (1) and (2) of assumption C are satisfied and such that R is also satisfied, and (b) in S , on the same line l in the same net $R(H_0H_1H_\infty)$ there is associated with every open cut (A, B) (with respect to the scale H_0, H_1, H_∞) a point $\bar{P}_{(A, B)}$ on l satisfying the same conditions (1), (2) and $\bar{R}$, then, for every (A, B) , $\bar{P}_{(A, B)}$ is identical with $P_{(A, B)}$.

This uniqueness of choice of the association in question does not however exist for the case of a complex space satisfying $A, E, H, C, \bar{R}$ and I . Hence it is not determined in advance

just what point must be associated with each cut. Indeed the reviewer believes that in this case for a given H_0, H_1, H_∞ on a given line l and a given open cut $(\bar{A}, \bar{B})$ on l not related in a certain way to H_0, H_1 , and H_∞ , one may select, for the $P_{(\bar{A}, \bar{B})}$ to be associated with $(\bar{A}, \bar{B})$ *any point whatsoever* of the line l with the exception of an infinity of points which are (in a certain sense) related algebraically to the points H_0, H_1, H_∞ —provided of course that after this selection is made, the points to be assigned to the other open cuts are properly chosen. Thus there appears to be involved here an arbitrary element of a very pronounced character. If the author claims that in using assumption C for a complex space he does not introduce a new undefined idea, then the reviewer would like to ask the following questions.

Why do the authors of volume I say, on page 1 of that volume, "Since any defined element or relation must be defined in terms of other elements and relations, it is necessary that one or more of the elements *and** one or more of the relations between them remain entirely *undefined*†; otherwise a vicious circle is unavoidable" and why, on page 15 of volume I do they say "We consider a class (cf. § 2, page 2) the elements of which we call *points*, and certain undefined *classes of points* which we call *lines*"? Why not say merely, "We consider a class the elements of which we call *points*" and substitute the following in place of the treatment beginning with line 9 of page 16?

"Concerning points we now make the following assumption:

"ASSUMPTION A. With every two points X and Y there is associated a class of points XY such that

(1)‡ if A, B and C are points which do not belong to XY for any two points X and Y , and D and E ($D \neq E$) are points such that B, C, D belong to X_1Y_1 for some pair of points X_1 and Y_1 and C, A, E belong to X_2Y_2 for some pair of points X_2Y_2 then there is a point F such that A, B, F belong to some X_3Y_3 and D, E, F belong to some X_4Y_4 .

(2)§ If $X \neq Y$ there are at least three points in the class XY .

(3) There do not exist two distinct points X and Y such that every point is in the class XY ."

Etc.

Can any objection be made to this procedure which

* Italics are the reviewer's.

† Italics are the authors'.

‡ Cf. A₁. § Cf. E₁. Cf. E₁.

would not also be an objection to the employment of C as an assumption for complex geometry? If in using such an assumption as C the author does not introduce a new undefined relation, then is it not at least true that in almost (or quite) every case where a set of postulates has ever been constructed in terms of more than one undefined element one can construct a closely corresponding set in terms of only *one* undefined element in a manner similar to that employed in C and in the example that I have given above?

A rather puzzling question now arises. If it be granted that in employing C in the set $A, E, H, C, \bar{R}, I$, the author is really introducing a new undefined relation then is it or is it not true that this is also the case for the set A, E, H, C, R ? Or does the mere fact that the above-mentioned arbitrary element is present in the first case and absent in the second—does this fact alone afford sufficient grounds for concluding that the use of C introduces a new undefined relation in the first connection but not in the second?

Aside from the above considerations, assumption C leaves something to be desired from the standpoint of analysis. And what are we to understand is the relation between $C, \bar{R}$ and I ? Should $\bar{R}$ be stated as a *separate* assumption? It reads "On some line l not all points belong to the same chain." Here we have a postulate which is stated separately from C but in which there is used a term (chain) which has been defined (?) in terms of an "association" postulated in C . It seems to me the situation would have been made clearer if instead of stating $\bar{R}$ and I separately the author had incorporated in postulate C , after the statement of condition (2), something like the following:

"(3) If in terms of this association of points with cuts 'the chain defined by A, B, C ' is defined as indicated on pages 17 and 21 for any three collinear points A, B, C ; then there exists a line $\bar{l}$ such that (a) on $\bar{l}$ not all points belong to the same chain, (b) through a point P of any chain C of the line $\bar{l}$ and any point J on $\bar{l}$ but not in C there is not more than one chain on $\bar{l}$ which has no other point than P in common with C ."

Of course this strengthened postulate C is more complicated than the original. But does not the attempt to state $\bar{R}$ and I as separate assumptions serve to becloud the true state of affairs?

In an article in the *Annals of Mathematics*, volume 11 (1909), page 34, J. W. Young says "The notion of a chain has been fundamental in the synthetic introduction of imaginaries into geometry since the time of von Staudt. In the more recent work on the foundations of projective geometry it necessarily plays an important rôle. Pieri has indeed recently chosen it as one of the undefined elements in his set of assumptions for complex geometry. More recently Professor Veblen and I have given a set of assumptions for projective geometry in which point and an undefined class of points called a line are the only undefined elements and in which the chain is defined." It is perhaps superfluous to say that the reviewer does not agree with this statement. First there at least seems to be room for debate on the question whether Veblen and Young do not employ a third undefined element. Secondly I certainly do not feel that they have defined chain anywhere in volume I or volume II. If it is true that point and line are the only undefined elements here and that chain is defined, then it would have been possible to have gotten along with point as the only undefined element, both line and chain being defined. One could* even do this on the basis of the single axiom that the cardinal number of the set of all points is C . In this case there would be an arbitrary element involved in the "definitions" of both line and chain. On what grounds if any can one object to this procedure without objecting to that of Veblen and Young? If we are to allow an arbitrary element in connection with the notion of a chain on what grounds can we object to doing an entirely similar thing in connection with the notion of a line? Perhaps it may be contended that it is undesirable that this sort of arbitrariness should be associated with the notion of a line because this notion is so fundamental in projective geometry. In complex projective geometry the notion of a chain is also quite fundamental.† Indeed I am not sure but that it might be considered just as fundamental as that of a line. It seems to the reviewer that there is much to be said in favor of Pieri's treatment in which point, line and chain are undefined. The question arises however whether, from assumptions C , $\bar{R}$ and I and the results that the authors have established concerning them, one may not perhaps obtain *suggestions* for a well

* See below.

† Cf., indeed, the above quotation from one of the authors of Volume I.

analyzed set of postulates in terms of point, line and chain or in terms of point, line and order, but simpler than that of Pieri.

On pages 302, 303 and 304 occur three statements which for purposes of reference I will call I, II and III. These statements are as follows:

I. "Assumptions I-IX, XVII are categorical for the euclidean space, i.e., if two sets of objects $[P]$ and $[Q]$ satisfy the conditions laid down for points in the assumptions, there is a one-to-one reciprocal correspondence between $[P]$ and $[Q]$ such that the subsets called lines of $[P]$ correspond to the subsets called lines of $[Q]$. Thus the internal structure of a euclidean space is fully determined by assumptions I-IX, XVII."

II. "Assumptions I-X, XVII have a different rôle from X-XVI or XVIII-N in that they determine the set of objects (points, lines, etc.) which are presupposed by all the other assumptions. The choice of these assumptions is logically arbitrary. The choice of such sets of "assumptions" as X-XVI is not arbitrary, it must correspond to a properly chosen group of permutations of the objects determined by I-X, XVII."

III. "The point of view of the writer is that if X-XVI or XVIII-N are to be regarded as independent assumptions their independence is of a lower grade than that of I-IX, XVII. They constitute a definition by postulates of a relation (congruence or nearness) among objects (points, lines, etc.) already fully determined."

As to I, what is meant by the "internal structure" of a euclidean space? Do not circles have as much to do with the internal structure of euclidean space as do straight lines? If so it seems to me that the internal structure of a euclidean space is not at all fully determined by assumptions I-X, XVII. What does it mean to say that two sets of objects $[P]$ and $[Q]$ satisfy the conditions laid down for points in the assumptions? The assumptions in question are in terms of point and order (not point alone). In order to give a definite interpretation of a space satisfying I-IX, XVII do we not need to be told just what objects are points for that interpretation and just what points A , B and C are in the order ABC in that interpretation? Grant that we are given such an interpretation S of a space satisfying assumptions I-X, XVII. Then for any three points it is fully determined

whether or not they are on the same line* and to that extent the internal structure of the space S is fully determined. But if one who is curious to know whether the interpretation in question is an interpretation of euclidean geometry should pick out three non-collinear points and ask "are these three points on a circle?"—What answer could be made? One might reply (?) "if you pick out a polar system at infinity I will tell you whether A , B and C are on a circle with respect to that polar system"—but would that be an answer to the question?

As to II and III, if the mere fact that a certain set of postulates is categorical in terms of something, no matter what (I suppose the statement "they determine the set of objects," etc., means that I–X, XVII are categorical in terms of point and order), implies that no further postulates are necessary for any purpose or at least that any further postulates are independent to a lesser degree, if at all, then why not base all of euclidean geometry on the following axiom?

AXIOM A. *The number of points in space is C (the power of the continuum).*

On the basis of Axiom A, straight lines, order and congruence can be so defined that Axioms I–IX of § 29 and Axioms† X–XVI of § 66 will all be fulfilled. Thus Axiom A would be a sufficient basis for three-dimensional euclidean geometry. But straight lines, order and congruence could also be defined in some other way so that all the theorems of two-dimensional euclidean geometry would be fulfilled. With the use of another set of definitions Bolyai-Lobachevskian geometry could be obtained, etc., etc.

With reference to the last two sentences of Statement II, —in what sense, if any, are assumptions I–X, XVII arbitrary? Must not one select II so that it will not contradict I or follow from I, select III so that it will not contradict I or II or follow from them, etc.? Is anything different true of X–XVI? Are they not arbitrary except in that they must be selected so as not to contradict each other or I–X, XVII, and so that no one of them will follow from the others together with I–X, XVII? In a certain sense it may of course be said

* In § 29 the notion of a line is defined in terms of point and order.

† Cf. "Foundations of Geometry," by Oswald Veblen, in *Monographs on Modern Mathematics*, edited by J. W. A. Young, New York, 1911. Also R. L. Moore, "Sets of metrical hypotheses for geometry," *Trans. Amer. Math. Society*, vol. 9 (1908), pp. 487–512.

that every time a new (independent) assumption is added there is a decrease in the arbitrariness involved in the selection of an additional assumption (unless one is to have either dependence or a contradiction) and after one has postulated I-X, XVII there is of course less arbitrariness than there was at the beginning. It is true that no assumption can be added to I-X, XVII (without there being a contradiction) if it gives additional information (of a certain type*) concerning point and order alone. But it is allowable to add assumptions that give other information.

On page 71 Veblen narrows a definition of Klein by "assigning to the geometry corresponding to a given group only the theory of those properties which, while invariant under this group, are *not invariant under any other group of projective collineations containing it.*" He adds "This will render the question definite as to whether a given theorem belongs to a given geometry." Unless the reviewer fails to understand the meaning of these statements it appears that according to this test it is not a theorem of euclidean geometry that in a given plane there is only one line parallel to a given line through a point not on that line, nor is it a theorem of euclidean geometry that through two given points there is only one line, and it is not a proposition of double elliptic geometry that every two coplanar lines have two points in common. Apparently the author himself does not always hold to this point of view. For instance on page 70 he says "We have thus considered only very general properties of figures and so have dealt hardly at all with the familiar relations, such as perpendicularity, *parallelism*,† congruence of angles and segments, which make up the bulk of elementary euclidean geometry."

On page 83 in order to secure a complete definition of a planar field of vectors should not the author add to (1) and (2) the stipulations that (3) for each vector V and point A there is only one point B such that V corresponds to the ordered point pair AB and (4) if two ordered point pairs AB and $A'B'$ are not equivalent under the group of translations then their corresponding vectors are distinct? The definition as it stands seems to be satisfied if the number 1 is taken as the vector of *every* point pair.

The present review has been much concerned with a dis-

* Cf. E. H. Moore, "On the foundation of mathematics," *Science*, vol. 17 (1903), 401-416.

† Italics are the reviewer's.

cussion of certain more or less debatable and delicate questions relating to the foundations of mathematics. Let it be not imagined that these questions bear in any way on the usefulness and interest of the main body of the treatise under review.

In Chapter II the author considers order relations in real projective space. Chapter III is concerned with affine plane geometry, that is to say with the geometry corresponding to the group of all those projective collineations which transform into itself the set of all points not lying on a given line l_∞ of a given projective plane π . One of the most interesting features of the text under review is the way in which various propositions are classified under particular groups. For instance, though of course the affine group doesn't leave all metrical properties invariant it does leave certain particular ones invariant and in Chapter III the author considers a considerable body of such properties. In particular it is interesting that a theory of equivalence of triangles can be based* on this group.

Chapter IV is concerned with euclidean plane geometry regarded as the geometry corresponding to the group of all those projective collineations that leave invariant a fixed involution (without double points) on a line l_∞ in a real projective plane. Chapter V is concerned with ordinal and metrical properties of conics, Chapter VI with inversion geometry and related topics, including complex chains, Chapter VII with affine and euclidean geometries of three dimensions and Chapter VIII with non-euclidean geometries.

Chapter IX is concerned with theorems on sense and separation, a large body of such theorems being proved on the basis of A, E, S and P without use of C and R . Sense classes of various sorts are defined in terms of elementary transformations. In euclidean space of two dimensions for an ordered set of three non-collinear points an elementary transformation is defined as the operation of replacing one of the three points in question by a point which is joined to it by a segment not meeting the line on the other two. In § 181 it is stated that the notion of right and left-handedness can be extended to curves by a limiting process. For a treatment of sense on curves in two dimensions without the use of such a limiting process reference may be made to an article by J. R. Kline.†

* Cf. page 96. Also footnote reference to Wilson and Lewis.

† J. R. Kline, "A definition of sense on closed curves in non-metrical analysis situs," *Annals of Mathematics*, vol. 19 (1918), pp. 185-200.

The reviewer feels that the volume under review is a valuable addition to the as yet rather restricted list of advanced mathematical treatises of high grade published in America.

R. L. MOORE.

NOTES.

At a special meeting on April 23, 1920, the Council of the American Mathematical Society approved the formation of an American section of the international mathematical union and authorized its committee on the union to take the necessary steps to organize the section. The Council also adopted a resolution that the publication of a journal of mathematical abstracts is very desirable, and authorized its committee on bibliography to take steps toward securing the financial support necessary for such a journal. It was agreed that the representatives of the Society in the division of physical sciences of the National research council should present these projects before the division. Accordingly, at its meetings on April 28-29, 1920, the division adopted resolutions recommending to the National research council that the American section of mathematics organized under the auspices of the division be made the authorized agent of the council in the organization of the proposed international mathematical union, and its representative in that body when organized.

At the same meetings the Mathematical association of America was given the right to nominate one member of the division. The number of members at large was increased by one and Professor G. D. BIRKHOFF was elected as the additional member. Professor OSWALD VELEN was elected a member of the executive committee of the division.

The project for the publication of a journal of mathematical abstracts was approved and a committee consisting of Professors L. E. DICKSON, OSWALD VELEN, and H. S. WHITE was appointed to work out details and consult with the finance committee of the council as to securing the necessary funds.

A committee was also appointed to secure a revolving fund for the publication of important scientific books and papers commercially unattractive to regular publishing houses. Provision was made for the appointment of research committees

on algebraic numbers and statistics. The division approved a plan for establishing research fellowships in mathematics, to be administered by the division.

THE April number (volume 21, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "Differential equations containing arbitrary functions," by G. A. BLISS; "Functions of lines in ballistics," by G. A. BLISS; "On the summability of the developments in Bessel's functions," by C. N. MOORE; "One-parameter families and nets of ruled surfaces and a new theory of congruences," by E. J. WILCZYNSKI; "Nets of space curves," by G. M. GREEN; "A set of postulates for fields," by N. WIENER; "A theorem on modular covariants," by O. C. HAZLETT.

THE March number (volume 21, number 3) of the *Annals of Mathematics* contains: "A Green's theorem in terms of Lebesgue integrals," by H. E. BRAY; "Bilinear operations generating all operations in a rational domain Ω ," by NORBERT WIENER; "On the enumeration of proper and improper representations in homogeneous forms," by E. T. BELL; "A proof of Jordan's theorem about a simple closed curve," by J. W. ALEXANDER; "Linear order in three dimensional euclidean and double elliptic spaces," by G. H. HALLETT, JR.; "Further properties of the general integral," by P. J. DANIELL; "Summability of double series," by L. L. SMAIL; "The fundamental theorem of celestial mechanics," by J. L. COOLIDGE,

AT the meeting of the National Academy of Sciences held at Washington, April 27, the following mathematical papers were read: By Professor L. E. DICKSON, "Recent notable progress in the theory of numbers"; by Professor EDWARD KASNER, "Geodesics and relativity." Professor H. F. BLICHFELDT, of Stanford University, has been elected a member of the academy, and Professor CAMILLE JORDAN, of the Collège de France, a foreign associate.

THE following university courses in mathematics are announced:

COLUMBIA UNIVERSITY. The list of courses announced for

the academic year 1920-1921 in the May BULLETIN should, have included the following:—By Professor C. J. KEYSER: Philosophy of mathematics, four hours.

CORNELL UNIVERSITY (academic year 1920-1921).—By Professor J. H. TANNER: Mathematics of finance, two hours.—By Professor VIRGIL SNYDER: Algebraic geometry, three hours; Symposium of mathematics, two hours.—By Professor F. R. SHARPE: Fourier series and the potential theory, three hours (first term).—By Professor W. B. CARVER: Metric geometry, three hours.—By Professor ARTHUR RANUM: Theory of numbers, three hours (second term).—By Professor D. C. GILLESPIE: Differential equations, three hours.—By Professor W. A. HURWITZ: Differential equations of mathematical physics, three hours.—By Professor C. F. CRAIG: Theory of functions of a complex variable, three hours.—By Professor F. W. OWENS: Advanced calculus, three hours.—By Dr. F. W. REED: Celestial mechanics, three hours.—By Dr. H. B. OWENS: Projective geometry, three hours.—By Dr. G. M. ROBISON: Infinite series, three hours.—By Mr. H. S. VANDIVER: Modern algebra and theory of equations, three hours.—By Dr. H. C. M. MORSE: Differential geometry, three hours.

HARVARD UNIVERSITY (academic year 1920-1921).—All courses meet three times a week throughout the year, except those marked *, which meet for half a year.—By Professor W. F. OSGOOD: Dynamics (second course); The theory of functions (second course, part I): functions on an algebraic configuration*.—By Professor C. L. BOUTON: Introduction to modern geometry and algebra; Geometrical transformations, with special reference to the work of Sophus Lie.—By Professor C. L. COOLIDGE: Probability*; Properties of polynomials and invariants*; Line geometry.—By Professor E. V. HUNTINGTON: The fundamental concepts of mathematics*.—By Professor O. D. KELLOGG: Introduction to the theory of potential functions and Laplace's equation*.—By Professor D. R. CURTISS (of Northwestern University): The analytical theory of heat and problems in elastic vibrations*; The theory of functions (second course, part II): functions defined by linear differential equations of the second order*.—By Professor G. D. BIRKHOFF: Differential and integral calculus

(advanced course); Developments in series*; Difference equations*.—By Professor W. C. GRAUSTEIN: The theory of functions (introductory course); Differential geometry of curves and surfaces.

Professors KELLOGG and BIRKHOFF will conduct a fortnightly seminar in analysis. Courses of research are also offered by Professor OSGOOD in the theory of functions, by Professor BOUTON in the theory of point transformations, by Professor COOLIDGE in geometry, by Professor KELLOGG in the theory of potential functions, by Professor BIRKHOFF in the theory of differential equations, and by Professor GRAUSTEIN in geometry.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY (summer session).—By Professor F. S. WOODS: Analytic geometry.—By Mr. L. H. RICE: Elements of differential equations.—By Professors C. L. E. MOORE, H. B. PHILLIPS, F. L. HITCHCOCK, and DOUGLASS: Theoretical mechanics (introductory course).

UNIVERSITY OF CALIFORNIA (academic year 1920-1921).—By Professor M. W. HASKELL: Higher plane curves, three hours.—By Professor D. N. LEHMER: Geometry of four dimensions, three hours (first term).—By Professor FLORIAN CAJORI: History of mathematics, two hours; Teachers' course, three hours (first term).—By Professor T. M. PUTNAM: Partial differential equations, three hours (first term); Special analytic functions, three hours (second term).—By Professor W. D. McDONALD: Functions of a complex variable, three hours.—By Professor B. A. BERNSTEIN: Logic of mathematics, two hours.—By Professor FRANK IRWIN: Introduction to higher algebra, three hours.—By Dr. PAULINE SPERRY: Differential geometry, three hours (second term).

UNIVERSITY OF WISCONSIN (summer session).—By Professor E. P. LANE: Differential equations, five hours; Modern analytic geometry, three hours.—By Professor H. W. MARCH: Mechanics, five hours; Differential equations of mathematical physics, five hours.—By Professor E. B. SKINNER: Differential geometry, five hours; Special topics in algebra, three hours.—By Professor ARNOLD DRESDEN: Elliptic integrals, five hours; Theory of point sets, five hours.

UNIVERSITY OF STRASBOURG.—The reorganized University of Strasbourg announces that it is now in a position to offer a complete course of research for the doctorate in mathematics and mathematical physics, as well as the usual introductory courses in analysis, mechanics and astronomy. The following research courses will be given during the academic year 1920–1921: First semester (November 1, 1920–February 28, 1921).—By Professor BAUER: Quanta theory and the structure of atoms, three hours.—By Professor FRÉCHET: Theory of chance, two hours; Integral equations, one hour. Second semester (March 1, 1921–June 30, 1921).—By Professor BAUER: Statistical applications of quanta theory, three hours.—By Professor FRÉCHET: Applications of the theory of chance, one hour; Functions of lines, two hours.—By Professor VILLAT: The motion of a solid in a viscous fluid, two hours.—By Professor PÉRÈS: Transformations of surfaces applicable to quadrics, two hours.—By Professor VALIRON: Dirichlet's series and factorial series, two hours.

Further information with regard to mathematics courses may be obtained from M. le Directeur de l'Institut de Mathématiques de l'Université, Strasbourg, Bas-Rhin, France, and details concerning lodgings, etc., and courses in the French language for foreign students given during the summer of 1920 by the Faculty of letters, from the Comité de Patronage des Etudiants étrangers, Université, Strasbourg.

PROFESSOR PAUL APPELL, honorary dean of the faculty of sciences of the University of Paris, has been made rector of the Paris academy, as successor to the late LUCIEN POINCARÉ.

SIR THOMAS MUIR has recently presented his collection of about 2,500 mathematical books and pamphlets to the South African Public Library at Capetown. It consists largely of sets of periodicals, some of them now very rare, and of a special library on the theory of determinants and allied subjects that is probably the most complete in existence.

PROFESSOR D. R. CURTISS, of Northwestern University, has been granted leave of absence to accept the position of lecturer in mathematics at Harvard University during the coming academic half-year.

PROFESSOR L. E. DICKSON, of the University of Chicago, has been elected a member of the American Philosophical Society.

ASSISTANT professor S. LEFSCHETZ, of the University of Kansas, has been promoted to an associate professorship of mathematics.

At the University of Oklahoma, associate professor E. P. R. DUVAL has resigned, and Mr. E. E. COWAN has been appointed instructor in mathematics.

At Adelbert College, Western Reserve University, Dr. W. G. SIMON has been promoted to an assistant professorship of mathematics. Dr. C. A. NELSON, of the University of Kansas, has been appointed instructor in mathematics.

ASSISTANT professor J. V. MCKELVEY, of Iowa State College, has been promoted to an associate professorship of mathematics.

DR. I. A. BARNETT, Benjamin Peirce instructor at Harvard University during the academic year 1919-1920, and Mr. H. R. BRAHANA, of Princeton University, have been appointed instructors in mathematics at the University of Illinois.

DR. C. N. REYNOLDS, of Wesleyan University, has been appointed instructor in mathematics at Dartmouth College.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

CASTELNUOVO (G.). *Lezioni di geometria analitica. Puntata 1. 4a edizione.* Milano-Roma-Napoli, soc. ed. Dante Alighieri, di Albrighi, Segati e C. (Città di Castello, S. Lapi), 1919. 8vo. 8 + 320 pp. L. 30.00

FOWLER (R. H.). *The elementary differential geometry of plane curves.* (Cambridge Mathematical Tracts, No. 20.) Cambridge, University Press, 1920. Demy 8vo. 7 + 105 pp. 6s.

GANGULI (S.). *Lectures on the theory of plane curves delivered to post-graduate students in the University of Calcutta. Parts 1 and 2.* Calcutta, University of Calcutta, 1919. 10 + 140 + 14 + 212 pp. + 16 plates.

SHUGERT (S. P.). The resolvents of König and other types of symmetric functions. (Diss., University of Pennsylvania.) Lancaster, Pa., 1919. 19 pp.

ZOLESE (C.). *Matematica pura o matematica applicata?* Discorso, il 30 aprile, 1919. Città di Castello, tip. Grifani-Donati, 1919. 8vo. 21 pp.

II. ELEMENTARY MATHEMATICS.

ASCOLI (G.). Complementi di geometria per gli istituti tecnici. 4a edizione. Livorno, Giusti, 1920. 8vo. 10 + 237 pp. L. 5.40

BORTOLOTTI (E.). *Aritmetica generale ed algebra per i licei classici e moderni.* Volume 1, per la prima classe. Milano-Roma-Napoli, soc. ed. Dante Alighieri, di Albrighi, Segati e C. (Città di Castello, S. Lapi), 1919. 8vo. 197 pp. L. 3.00

BURALI-FORTI (C.). See Pensa (A.).

BURALI-FORTI (C.) e RAMORINO (A.). *Elementi di algebra per le scuole medie inferiori.* 6a edizione. Torino, casa ed. G. B. Petrini, di G. Gallizio (V. Bona), 1920. 8vo. 7 + 142 pp. L. 3.00

CATANIA (S.). *Elementi di aritmetica ed algebra per le scuole secondarie inferiori.* 7a edizione, migliorata e semplificata, e adattata pure ai nuovi proposti programmi. Catania, N. Giannotta (V. Giannotta), 1919. 16mo. 8 + 382 pp. L. 4.20

JENKINS (F.). See Lennès (N. J.).

LENNÈS (N. J.) and JENKINS (F.). *Applied arithmetic, the three essentials.* Books 2 and 3. (Lippincott's School Text Series.) Philadelphia, Lippincott, 1920. 9 + 294 + 9 + 340 pp. \$0.80 + 0.88

PALATINI (F.). *Algebra, ad uso delle scuole medie superiori.* Parte 1-2. 5a edizione. Torino, casa ed. G. B. Petrini, di G. Gallizio (tip. Elzeviriana, Olivero e C.), 1920. 8vo. 76 + 126 pp. L. 6.50

PEANO (G.). See TAVOLE.

PENSA (A.). *Elementi di geometria ad uso delle scuole secondarie inferiori.* Prefazione di C. Burali-Forti. 6a edizione. Torino, casa ed. G. B. Petrini, di G. Gallizio (V. Bona), 1919. 8vo. 11 + 198 pp. con due prospetti. L. 4.00

RAMORINO (A.). See Burali-Forti (C.).

REEVE (W. B.). See Schorling (R.).

SCHORLING (R.) and REEVE (W. B.). *General mathematics.* Boston, Ginn, 1919. 16 + 488 pp. \$1.48

STRAZZERI (V.). *Geometria elementare.* Palermo, D. Capozzi, 1919. 8vo. 8 + 272 pp. L. 5.00

TAVOLE numeriche (quadrato, cubo, radice quadrata e cubica, logaritmo, reciproco, seno, coseno, tangente, cotangente, logaritmi naturali, tavole d'interesse, ecc.) con prefazione e note di Giuseppe Peano. Torino, Unione tipografico-editrice, 1920. 16mo. 35 pp.

TESTI (G. M.). *Corso di matematiche, ad uso delle scuole secondarie superiori e più specialmente degli istituti tecnici.* Volume 5: *Complementi d'algebra.* 3a edizione, corretta ed aumentata, coll'aggiunta di un'appendice sulla funzione derivata e sugli integrali. Livorno, Giusti, 1920. 8vo. 9 + 330 pp. L. 5.00

- . Elementi d'algebra, ad uso più specialmente dei licenziandi delle scuole tecniche. 16a edizione riveduta. Livorno, Giusti, 1920. 16mo. 6 + 86 pp. L. 1.80
- . Elementi di geometria, ad uso degli alunni delle scuole tecniche. 23a edizione riveduta. Livorno, Giusti, 1920. 16mo. 11 + 239 pp. L. 4.00
- . Elementi di matematica, ad uso degli alunni delle scuole normali. Fasc. 3, per la 2a classe. 6a edizione riveduta. Livorno, Giusti, 1920. 16mo. 8 + 107 pp. L. 2.00

III. APPLIED MATHEMATICS.

- BRAHE (T.). Tychonis Brahe Dani opera omnia. Edidit J. L. E. Dreyer. Tomus 6. Hauniae, Libreria Gyldendaliana, 1920. 5 + 375 pp.
- BROSE (H. L.). See FREUNDLICH (E.).
- DECKERT (A.). Mechanik. Kempten, 1919.
- DELSOL (E.). Considérations sur les unités fondamentales de la physique. Paris, chez l'auteur, 1919. 8vo. 31 pp.
- DORGEOT (E.). Cinématique théorique et appliquée. Paris, Dunod et Pinat, 1919. 4to. 272 pp. Fr. 54.00
- DREYER (J. L. E.). See BRAHE (T.).
- EINSTEIN (A.). See FREUNDLICH (E.).
- FILE (L. K.). See MOIR (H.).
- FREUNDLICH (E.). The foundations of Einstein's theory of gravitation. Authorized English translation by H. L. Brose. With a preface by A. Einstein and an introduction by H. H. Turner. Cambridge, University Press, 1920. Demy 8vo. 16 + 61 pp. 5s.
- GARNERI (A.). Corso elementare di disegno geometrico, ad uso delle scuole medie, complementari, professionali, ecc. Parte 2: Proiezioni ortogonali con applicazioni alla meccanica e alla teoria delle ombre prospettiva parallela. 9a edizione, riveduta, corretta, aumentata. Torino, ditta G. B. Paravia e C. (Rocca S. Casciano, L. Capelli), 1919. 16mo. 8 + 180 pp. L. 3.00
- GIBBS (W.). L'équilibre des substances hétérogènes. Exposé abrégé, traduit et complété de notes explicatives par Georges Matisse. Paris, Gauthier-Villars, 1919. 16mo. 8 + 102 pp. Fr. 5.25
- GOFFI (C.). L'apprendista meccanico. 2a edizione, riveduta et ampliata. (Manuali Hoepli.) Milano, Hoepli, 1920. 24mo. 19 + 381 pp. L. 7.50
- JUNG (H.). See NÖLKE (F.).
- MATISSE (G.). See GIBBS (W.).
- MOIR (H.), CRAIG (J. D.), FILE (L. K.), MACLEAN (A. T.) and WOLFENDEN (H. H.). Sources and characteristics of the principal mortality tables. (Actuarial Studies, No. 1.) New York, Actuarial Society of America, 1919. 4 + 79 pp. \$1.25
- TURNER (H. H.). See FREUNDLICH (E.).
- WILSON (E. B.). Aeronautics. A class text. New York, Wiley, 1920. 8 + 265 pp. \$4.00

Whole No. 289

CONTENTS

	PAGE
The Fourteenth Western Meeting of the American Mathematical Society. By Professor ARNOLD DRESDEN -	385
The April Meeting of the San Francisco Section. By Professor B. A. BERNSTEIN -	396
The Sum of the Face Angles of Certain Polyhedrons in n -Space. By Dr. H. F. MACNEISH -	398
On Bordered Fredholm Determinants. By Professor T. H. HILDEBRANDT -	400
On the Coherence of Certain Systems in General Analysis. By Professor A. D. PITCHER -	405
On the Adjoint of a Certain Mixed Equation. By Dr. R. F. BORDEN -	408
The Second Volume of Veblen and Young's Projective Geometry. By Professor R. L. MOORE -	412
Notes -	425
New Publications -	430

Changes of address of members, exchanges, and subscribers should be communicated at once to the Secretary of the American Mathematical Society, 501 West 116th Street, New York.

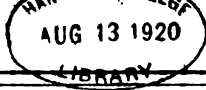
Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to The American Mathematical Society, 41 North Queen St., Lancaster, Pa., or 501 West 116th Street, New York.

The initiation fees and annual dues of members of the American Mathematical Society are payable to the Treasurer of the American Mathematical Society, Professor J. H. TANNER, Cornell University, Ithaca, N. Y.

Articles for insertion in the BULLETIN should be addressed to the
BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.

The following dates have been fixed for the meetings of the Society :

The Twenty-Seventh Summer Meeting and Ninth Colloquium of the Society will be held at the University of Chicago, extending through the week September 6-11, 1920.



BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY

**A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE**

EDITED BY

F. N. COLE	VIRGIL SNYDER	J. W. YOUNG
ALEXANDER ZIWET	D. E. SMITH	
T. LEVI-CIVITA	R. C. ARCHIBALD	

VOLUME XXVI., NUMBER 10

JULY, 1920

**PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK
1920**

Entered at the Post Office at Lancaster, Pa., as second-class matter
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER

Whole No. 290

\$5.00 a year

Publications of the American Mathematical Society

Transactions of the American Mathematical Society.
Published quarterly. Subscription price for annual
volume, \$5.00; to members of the Society, \$3.75.

Mathematical Papers of the Chicago Congress, 1893.
Price, \$3.00; to members, \$1.50.

Evanston Colloquium Lectures, 1893. By FELIX KLEIN.
Price, 75 cents.

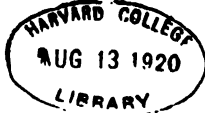
Boston Colloquium Lectures, 1903. By H. S. WHITE, F.
S. WOODS and E. B. VAN VLECK. Price, \$2.00; to
members, \$1.50.

Princeton Colloquium Lectures, 1909. By G. A. BLISS
and EDWARD KASNER. Price, \$1.50; to members
\$1.00.

Madison Colloquium Lectures, 1913. By L. E. DICKSON
and W. F. OSGOOD. Price, \$2.00; to members, \$1.50

Cambridge Colloquium Lectures, 1916. Part I. By G
C. EVANS. Price, \$1.50; to members, \$1.00.

Address all orders to AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.



THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY IN NEW YORK.

THE two hundred and tenth regular meeting of the Society was held at Columbia University on Saturday, April 24, 1920, extending through the usual morning and afternoon sessions. The total attendance numbered over one hundred and thirty, and included the following eighty-two members:

Dr. J. W. Alexander, Professor R. C. Archibald, Dr. I. A. Barnett, Professor F. W. Beal, Mr. D. R. Belcher, Professor Suzan R. Benedict, Professor A. A. Bennett, Professor W. J. Berry, Mr. William Betz, Professor Joseph Bowden, Professor R. W. Burgess, Professor B. H. Camp, Professor C. W. Cobb, Professor A. Cohen, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor E. S. Crawley, Dr. W. L. Crum, Professor Louise D. Cummings, Professor L. E. Dickson, Professor L. P. Eisenhart, Professor H. B. Fine, Professor T. S. Fiske, Professor W. B. Fite, Dr. L. R. Ford, Professor O. E. Glenn, Dr. T. H. Gronwall, Professor Olive C. Hazlett, Professor E. R. Hedrick, Dr. A. A. Himwich, Professor T. R. Hollcroft, Professor E. V. Huntington, Mr. S. A. Joffe, Professor Edward Kasner, Professor O. D. Kellogg, Professor C. J. Keyser, Dr. E. A. T. Kircher, Dr. K. W. Lamson, Mr. Harry Langman, Professor P. H. Linehan, Mr. L. L. Locke, Professor W. R. Longley, Mr. John McDonnell, Professor C. R. MacInnes, Professor H. P. Manning, Dr. A. L. Miller, Professor H. H. Mitchell, Professor R. L. Moore, Professor Frank Morley, Professor G. W. Mullins, Professor W. F. Osgood, Mr. George Paaswell, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor Susan M. Rambo, Professor H. W. Reddick, Professor L. W. Reid, Mr. L. H. Rice, Professor R. G. D. Richardson, Dr. J. F. Ritt, Professor J. E. Rowe, Professor Paul Saurel, Dr. Caroline E. Seely, Professor S. P. Shugert, Professor P. F. Smith, Professor C. M. Sparrow, Dr. J. M. Stetson, Mr. J. J. Tanzola, Professor J. I. Tracey, Mr. H. S. Vandiver, Professor E. B. Van Vleck, Professor Oswald Veblen, Mr. A. L. Wechsler, Mr. R. A. Wetzel, Professor H. S. White, Professor J. K. Whittemore, Miss Ella C. Williams, Professor F. B. Williams, Professor A. H. Wilson, Professor Ruth G. Wood, President R. S. Woodward.

President Frank Morley occupied the chair, yielding it to Ex-President R. S. Woodward during the presentation of the papers on relativity at the afternoon session. The Council announced the election of the following persons to membership in the Society: Professor H. S. Everett, Bucknell University; Dr. L. J. Rouse, University of Michigan; Professor Nilos Sakellariou, University of Athens; Mr. H. L. Smith, University of Wisconsin; Professor Eugene Taylor, University of Wisconsin; Professor W. P. Webber, University of Pittsburgh. Thirteen applications for membership in the Society were received.

Professor L. P. Eisenhart was reelected a member of the Editorial Committee of the *Transactions* for a term of three years beginning October 1, 1920. Professor P. F. Smith will retire from the Editorial Committee on October 1, after nine years' service as editor, and Professor G. D. Birkhoff will fill out Professor Smith's unexpired term. Professor Oswald Veblen was appointed representative of the Society in the Division of Physics of the National Research Council for a term of three years beginning July 1, 1920. Professor Veblen's Cambridge Colloquium Lectures on Analysis Situs will be published by the Society late in the fall. Committees were appointed to confer with a committee of the Mathematical Association as to joint plans for future meetings and to prepare nominations for officers and other members of the Council to be elected at the annual meeting in December.

On the recommendation of the Council it was unanimously voted to incorporate the Society under the membership corporations law of the state of New York. The new form of organization will involve hardly any changes beyond those necessary to comply with legal requirements. The Council will continue as at present constituted. But it is necessary under the law to have an elected Board of Trustees; this will consist of the officers and other members of the Council who are elected by the Society at the annual meetings. Otherwise the Constitution and By-Laws, which have come down almost from the beginnings of the Society and which are a highly efficient instrument of government, well worthy of study, will remain practically as at present.

The Council received a preliminary report from the committee on reorganization, which will make specific recommendations at a later meeting. A report was also received

from the committee on the International Mathematical Union, and the formation of an American Section of the Union was approved. The recommendation of the committee on bibliography that a journal of mathematical abstracts be established was approved and the committee was authorized to take steps toward securing the necessary financial support.

The greater part of the afternoon session was devoted to a symposium on the subject of relativity, at which the following papers were presented:

I. "The physical and philosophical significance of the principle of relativity and Einstein's theory of gravitation," by Professor LEIGH PAGE, of Yale University.

II. "Geometric aspects of the Einstein theory," by Professor L. P. EISENHART, Princeton University.

Over fifty members took luncheon together between the sessions, and thirty gathered at the dinner after the meeting.

The regular programme consisted of the following papers:

(1) Professor N. A. COURT: "On a pencil of nodal cubics. Second paper."

(2) Mr. E. L. POST: "Introduction to a general theory of elementary propositions."

(3) Mr. E. L. POST: "Determination of all closed systems of truth tables."

(4) Mr. JESSE DOUGLAS: "The dual of area and of volume."

(5) Professor J. K. WHITEMORE: "Reciprocity in a problem of relative maxima and minima."

(6) Dr. I. A. BARNETT: "Linear partial differential equations with a continuous infinitude of variables."

(7) Dr. I. A. BARNETT: "Functionals invariant under one-parameter continuous groups in the space of continuous functions."

(8) Professor T. R. HOLLCROFT: "A classification of plane involutions of order four."

(9) Dr. TOBIAS DANTZIG: "A group of line-to-line transformations."

(10) Dr. A. R. SCHWEITZER: "On the iterative properties of the abstract field."

(11) Dr. J. F. RITT: "On the conformal mapping of a region into a part of itself."

(12) Dr. L. R. FORD: "A theorem relative to rational approximations to irrational complex numbers."

(13) Professor L. E. DICKSON: "Report on recent progress in the theory of numbers."

(14) Professor G. D. BIRKHOFF: "Note on the ordinary linear differential equation of the second order."

(15) Professor JOSEPH LIPKA: "The motion of a particle on a surface under any positional forces."

(16) Professor JOSEPH LIPKA: "Note on velocity systems in a general curved space of n dimensions."

(17) Professor J. E. ROWE: "Testing the legitimacy of empirical equations by an analytical method."

(18) Professor OSWALD VEBLER: "Relations between certain matrices used in analysis situs."

(19) Professor O. D. KELLOGG: "A simple proof of a closure theorem for orthogonal function sets."

(20) Professor C. L. E. MOORE: "Rotation surfaces of constant curvature in a space of four dimensions."

(21) Mr. H. S. VANDIVER: "On Kummer's memoir of 1857 concerning Fermat's last theorem."

(22) Professor NILOS SAKELLARIOU: "A note on the theory of flexion."

Mr. Post was introduced by Professor Keyser; Mr. Douglas by Professor Kasner. The papers of Professor Court, Mr. Post (second paper), Dr. Barnett (first paper), Dr. Dantzig, Dr. Schweitzer, Dr. Ritt, Professor Birkhoff, Professor Lipka, Professor Rowe, Professor Veblen, Professor Kellogg, Professor Moore, Mr. Vandiver, and Professor Sakellariou were read by title.

Abstracts of the papers follow below, being numbered to correspond to the titles in the list above.

1. Continuing the discussion started in his first paper (see this BULLETIN, February, 1920) Professor Altshiller-Court proves that: Two nodal cubics having in common three collinear points, the double point, and the two tangents at this point, are homologous, the double point and the base of the three collinear points being the center and the axis of homology, respectively. Several additional properties of the pencil of nodal cubics referred to are derived from this proposition. As an interesting special case of the dual of this proposition is pointed out the following: Two tricuspidal hypocycloids tangent to three concurrent lines are similar and similarly placed, the point of intersection of the three common tangents being the homothetic center of the two curves.

2. In this paper Mr. Post studies in its entirety the deductive system which Whitehead and Russell have developed in Part I, Section A, of their *Principia Mathematica*. Through the concept of the truth table of a truth function, a uniform method is given for telling whether the assertion of a given propositional function of the system can or cannot be derived from the postulates. By means of this result, a number of properties of the system are obtained, among which is the theorem that any propositional function of the system can either be asserted by means of the postulates or else is inconsistent with them.

Two modes of generalizing the system are considered. One consists in generalizing the primitive functions by means of the truth table concept, and connects up with the work of Sheffer and Nicod. The second or postulational method of generalization is shown to introduce new logical systems.

3. Corresponding to each of the 2^n sets of truth values of the arguments of a truth function $f(p_1, p_2, \dots, p_n)$, there is a unique truth value of the function. The relation thus set up may be called the truth table of f . Mr. Post considers the systems of truth tables that can be generated by combining arbitrary primitive truth tables and shows that there are 66 different systems generated by primitive tables with no more than three arguments, and 8 infinite families of systems which require tables of four or more arguments. A formula is given for the tables in each system, and it is shown that they include all closed systems of truth tables. These results are applied to the determination of all the ways in which the logical system of truth functions may be generated by independent primitive functions.

4. The object of Mr. Douglas' paper is to develop for the geometry based on the (oriented) line as element a concept analogous to that of area for the geometry based on the point. The main results are as follows:

Denoting by ω , p Hessian line coordinates, the numerical measure of any continuous "mass" of (oriented) lines is equal to $\int \int d\omega dp$. The justification of this definition consists in the fact that the measure so defined possesses the two essential properties of additivity and independence of rigid motion.

The boundary lines of any continuous "mass" envelop, in the usual case, an ordinary closed curve. The double integral $\iint d\omega dp$ is equal to $\int p d\omega$ taken over this boundary, which last is equal to the length of the bounding curve. The two geometric magnitudes customarily associated with any closed curve, namely the area and the perimeter, thus appear as duals of one another.

By means of a representation of the lines of the plane upon the points of a cylinder as follows: the (oriented) line of Hessian coordinates ω, p corresponds to the point of longitude ω and altitude p , the notion of measure is then extended to non-continuous line sets.

In the latter part of the paper the analogous ideas for three-dimensional space are considered. The measure of a "mass" of planes is found to be $\iiint \cos \varphi d\theta d\varphi dp$, where θ, φ, p are polar plane coordinates. On interpreting this integral geometrically, the plane measure of a polyhedron turns out to be equal to $\frac{1}{2} \sum e_i A_i$, where e_i is the length of any edge and A_i the dihedral angle of the (oriented) planes meeting in that edge.

5. Professor Whittemore's paper considers the following questions: If $\varphi(x, y) = u$ and $\psi(x, y) = v$ are single-valued real functions of two real variables x, y , having continuous partial derivatives of the first and second orders in the neighborhood of $P(x_0, y_0)$, A: When is $\varphi(x_0, y_0) = u_0$ a maximum or a minimum of u with the condition $\psi = v_0$? B: When is v_0 a maximum or a minimum of v with the condition $\varphi = u_0$? If there are extremes in both A and B, when are they like extremes?

It is proved for the general case, where not both first derivatives of either φ or ψ vanish at P , that a necessary condition for an extreme for both A and B is that the two curves $\varphi = u_0$ and $\psi = v_0$ be tangent at P ; that a sufficient condition for an extreme in both problems is that these curves do not osculate at P ; when both these conditions are satisfied that the extremes in A and B are like or unlike as φ_y and ψ_y have unlike or like signs at P .

The exceptional case, where both first derivatives of φ or of ψ or of both functions vanish at P , is considered; necessary and sufficient conditions for a maximum or a minimum are obtained in so far as this is possible by use of the first and second deriva-

tives. In some cases the discussion fails to establish such conditions. For several cases it is shown that, in A, u_0 is a maximum or minimum as $\psi_{yy}H + 2\varphi_{yy}\Delta\psi$ is negative or positive at P , where

$$H = \varphi_{xx}\psi_{yy} + \psi_{xx}\varphi_{yy} - 2\varphi_{xy}\psi_{xy}, \quad \Delta\psi = \psi_{xy}^2 - \psi_{xx}\psi_{yy}.$$

In several of the cases where u_0 and v_0 are extremes in A and B respectively it is proved that they are like or unlike extremes as H is positive or negative.

The invariant character of the conditions established for any change of variables, not singular at P , is considered. Simple examples illustrating the theory are given.

6. In a previous paper Dr. Barnett has given an existence theorem for solutions of linear non-homogeneous partial differential equations with a continuous infinitude of unknowns. In the present paper the theory there developed is applied to linear homogeneous equations. It is found convenient to use the theory of implicit functional equations as developed by Lamson in his Chicago dissertation.

7. Dr. Barnett considers one-parameter continuous groups of transformations $\bar{u}(x) = F[x, u(\xi), t]$ where F is a functional operation yielding for each continuous function of a given range and for each value of the parameter t , another continuous function. Infinitesimal transformations are defined and invariants of the groups are considered. These invariant functionals $G[u(x)]$ are found to be solutions of certain functional equations given in the preceding paper. Examples which are analogues of some of the well-known groups in n -space are studied.

8. Professor Hollcroft discusses the conditions imposed upon two planes when an algebraic (1,4) point correspondence is established between them. The particular correspondence given is defined by a curve of order n with a point P of multiplicity $n - 4$ and a line pencil on P , and is one of ten independent types of plane involutions of order four. An independent type is one that cannot be reduced to another by any birational transformation.

9. In a previous communication to the Society (April, 1916,

Chicago Section) Dr. Dantzig considered certain properties of point-to-point transformations and particularly a certain system of ∞^2 conics which he called the indicating system of the transformation. These properties are here extended to line-to-line transformations, and a special study is made of those transformations which admit a focal line, i. e., a bearer of an invariant range of points. Especially interesting is the case where the line at infinity is a focal line. The indicating system then consists of parabolas. These transformations form a group, the parabolic group G . The fundamental property of a transformation belonging to G is the following: Let l and $\bar{l}$ be two corresponding lines meeting in a point λ ; let moreover τ and $\bar{\tau}$ be the points where the indicatrix I touches l and $\bar{l}$ respectively. If c and c' are any two curves touching l in p and p' , while their transforms $\bar{c}$ and $\bar{c}'$ touch $\bar{l}$ in $\bar{p}$ and $\bar{p}'$, then the ratio

$$\kappa = \frac{\overline{pp'}}{pp'} = \frac{\bar{\tau}\lambda}{\tau\lambda}$$

and consequently is invariant along the line l . The transformations which leave the ratio κ invariant throughout the plane form a subgroup S of G . However, the subclass of S admitting the same value of κ , say κ_0 , does not form a subgroup of S , unless $\kappa_0 = 1$. In this latter case we obtain the Scheffers equilog group.

The rest of the paper is taken up by the generalization of these properties to the case where the focal line is of general position and by the interpretation of the corresponding Scheffer's subgroup. There is also given a general determination of the parabolic transformation admitting an axis and a detailed study of the so-called Laguerre group of line-to-line transformations.

10. In a previous paper Dr. Schweitzer constructed a set of postulates (with explicit reference to iterative compositions) for an abstract field in terms of two undefined quasi-transitive functions, $f(x, y)$ [$x - y$] and $f_1(x, y)$ [x/y] corresponding to postulates for an abstract group in terms of a relation concretely interpreted by $x \cdot y^{-1}$. In another paper, primarily in reference to the abstract field, he considered quasi-transitive functional equations of the type

$$(1) \quad f\{\lambda f(x, y), \mu f(x, z)\} = \nu f(z, y).$$

A solution of (1) for $\lambda(x) = \mu(x)$, $\nu(x) = x$ is

$$f(x, y) = X^{-1}\{c'X(x)/X(y) + c''\}, \quad \lambda(x) = X^{-1}\{X(x) - c''\},$$

where c' and c'' are constants. Second, a solution of (1) for $\lambda(x) = \mu(x) = x$ is $f(x, y) = X^{-1}\{cX(x) - cX(y)\}$, $\nu(x) = X^{-1}\{cX(x)\}$. If the "arbitrary" constant c is interpreted as a parameter, say t , then special cases of the latter function are

$$f_1(x, y, t_1) = (x - y) \cdot t_1, \quad f_2(x, y, t_2) = (x - y)/t_2,$$

each of which may serve abstractly as a ternary generating relation for an abstract field.*

11. Let a domain composed of a continuum plus its frontier be mapped conformally, in a one-to-one manner, into a part of itself, so that every frontier point of the original domain becomes an interior point. Dr. Ritt shows that, under the transformation, one point and only one point stays fixed. The derivative of the mapping function is less than unity in modulus at the fixed point. If the transformation is iterated, the consequents of every point of the original domain approach the fixed point. For the case of a simply connected domain this theorem has already been given by Julia, without assuming that the transformation is one-to-one. Julia's method, which is entirely different from that used in the present paper, does not admit of extension to a general continuum.

12. Let ω be any irrational complex number, and consider the inequality

$$\left| \omega - \frac{p}{q} \right| < \frac{k}{|q|^2},$$

where k is real and p/q is a rational fraction (i. e., p and q are each of the form $m + ni$, where m and n are real integers). Dr. Ford proves the following theorem: If $k = 1/\sqrt{3}$ there are infinitely many fractions satisfying the given inequality; if $k < 1/\sqrt{3}$ there exists an everywhere dense set of ω 's for each of which the inequality holds for only a finite number of fractions.

* Cf. this BULLETIN, June, 1918, p. 428; March, 1915, p. 295; October, 1915, p. 4; October, 1914, p. 28; Nov., 1918, pp. 58, 59.

13. Professor Dickson made a report on the literature on the theory of numbers during the past decade, prepared in accordance with a decision by the members of the Division of Physical Sciences of the National Research Council to prepare reports of recent progress in various branches of astronomy, mathematics, and physics. While most of these reports will probably cover only the past year, it seemed best in the case of a mathematical subject to cover the past decade, especially in view of the present difficulty of obtaining complete references in a field of mathematics. The present task was simplified by the fact that the writer had already taken great pains to cover the literature to date in his *History of the Theory of Numbers*, now in course of publication. During the past decade there have appeared fully 1,600 papers, notes and books on this subject. Those relating to divisibility and primality have been considered in volume I of the *History*, and are only cited en masse in this report in connection with the still more recent papers. In the case of Diophantine analysis (the subject of volume II, in press), only the more important recent papers are cited in this report. But there is given a rather complete list of recent papers on the remaining topics, since their appearance in volume III will be delayed at least a year.

14. Professor Birkhoff shows that it is possible to develop the theory of the ordinary linear differential equation of the second order with real independent variable in such wise that the conditions imposed are both necessary and sufficient throughout. In the usual formulation of the theory the conditions used are not of this type.

15. The complete system of trajectories of a particle constrained to move on a surface under any positional forces form a family of ∞^2 curves, one through each point in each direction for each value of the speed. Professor Lipka derives some of the geometric properties of such a system of curves. One of the simplest of these properties may be stated as follows: in each direction through a given point there passes one trajectory which hyperosculates its circle of constant geodesic curvature; the locus of the centers of geodesic curvature of the ∞^1 hyperosculating trajectories obtained by varying the direction through the point is a conic passing through the

given point in the direction of the force acting at that point. If the force is conservative, the complete systems of brachistochrones and catenaries on a surface also form families of ∞^3 curves. Some properties of these systems are derived, and a comparative study of the three systems, trajectories, brachistochrones and catenaries, is made.

16. In a previous paper,* Professor Lipka characterized certain systems of curves, termed velocity systems, in a general curved space of n dimensions, by means of their geometric properties. In the present note, the author points out a dynamical problem which leads to the same system of curves.

17. After the form of an empirical equation is assumed, the one equation of this assumed type which satisfies the data with least error may be obtained. But the question which remains unanswered is this: Have the data been forced to satisfy an equation of this particular type by virtue of the assumed form of the equation, or do they satisfy an equation of this type naturally? In Professor Rowe's paper an analytical method is developed which enables us to ascertain what type of equation the data (which are known to be correct) most naturally satisfy. The theory involved is a very great addition to, and is not in any sense a substitution for, any good judgment that might be used originally in assuming the most probable type of equation.

18. In Professor Veblen's paper, the matrix giving the number of intersections, counted with regard to sign, of a complete set of i -dimensional manifolds in an n -dimensional manifold, M_n , with a complete set of $(n-i)$ -dimensional manifolds of M_n is called an intersection matrix. It is shown how to obtain these matrices from the matrices giving the relations between the oriented j -cells and the oriented $(j-1)$ -cells on their boundaries by means of algebraic operations.

19. Let $n(x)$ be the set of normal functions arising from a differential equation of the Sturm type with general self-adjoint homogeneous linear boundary conditions. Professor

* "Natural families of curves in a general curved space of n dimensions," *Trans. Amer. Math. Society*, vol. 13, pp. 77-95.

Kellogg's note gives a simple proof that if this set is closed with respect to continuous functions, it is also closed with respect to summable functions which are not null functions. Hilbert and others have shown that they are closed with respect to continuous functions, and they are therefore closed with respect to the broader class. The interest of the note lies rather in the method of proof than in the results, which are largely already known.

20. In this note Professor Moore discusses the forms of curves that will generate surfaces of constant curvature when rotated by the special rotations leaving a doubly infinite number of planes invariant.

21. In an article in the *Mathematische Abhandlungen* of the Berlin Academy for the year 1857, pages 41-74, Kummer essayed to prove that the relation

$$(1) \quad x^p + y^p + z^p = 0$$

could not be satisfied in integers, when p is an odd prime not satisfying three given conditions. Based on this result, the conclusion that (1) is impossible for all p 's less than 100 was derived by him. In the present paper Mr. Vandiver points out that Kummer made several errors in his argument, which vitiate his results. The paper will appear in the *Proceedings of the National Academy of Sciences*.

F. N. COLE,
Secretary.

STIELTJES DERIVATIVES.

BY PROFESSOR P. J. DANIELL.

THE fundamental theorem for the derivative with respect to a function of limited variation is difficult to prove in the case of several dimensions, and no attempt is made here to consider the most general derived numbers. In place of the method used by the author for one dimension we shall use methods and ideas due to C. de la Vallée Poussin and W. H. Young.*

* C. de la Vallée Poussin, *Intégrales de Lebesgue*. Paris, 1916, pp. 61-73. W. H. Young, *Proc. London Math. Society* (1914), vol. 13, p. 109. P. J. Daniell, *Transactions Amer. Math. Society* (1918), vol. 19, p. 353.

The theorem to be proved in this paper is:

If in a fundamental interval of several dimensions $\beta(e)$, $F(e)$ are additive functions of sets, $\beta(e)$ non-negative and $F(e)$ absolutely continuous with respect to $\beta(e)$, then there exists a derivative $D_\beta F$ of F with respect to β everywhere except on a set e for which $\beta(e) = 0$; and if E is any set (measurable Borel) contained in the interval,

$$F(E) = \int_E (D_\beta F) d\beta(e).$$

The derivative used is not that obtained from sequences of intervals independent for each point, but a net-derivative.

General considerations. The points p are specified by n coordinates x_r ($r = 1, 2, \dots, n$) and an interval consists of a collection of points such that each coordinate lies in a linear continuous interval. In a one-to-one transformation which leaves relative order unchanged, $\beta(e)$, $F(e)$ and $D_\beta F$ will be absolutely invariant, so that the given interval, whatever it may be, can be transformed into one which lies "strictly" within the interval $x_r = 0$ to 1 ($r = 1, 2, \dots, n$). By defining $\beta(e)$, $F(e)$ to be 0 for sets not in the transformed interval, we can finally use for the fundamental interval one closed on the left, open on the right, namely

$$0 \leq x_r < 1 \quad (r = 1, 2, \dots, n).$$

$\beta(e)$, $F(e)$ are supposed to be additive in the sense of C. de la Vallée Poussin, that is to say, additive for a countably infinite as well as for a finite number of sets without common points.

Definition 1. If $\beta(e)$, $F(e)$ are additive functions of sets, $F(e)$ is said to be absolutely continuous with respect to $\beta(e)$, if $F(e) = 0$ for all sets e for which $\beta(e) = 0$.

Divide the interval $0 \leq x_r < 1$ ($r = 1, 2, \dots, n$) into 2^{ni} equal subintervals (of order i). These will be of the type

$$(m_r - 1)2^{-i} \leq x_r < m_r 2^{-i} \quad (r = 1, 2, \dots, n),$$

where, for each r , m_r is an integer between 1 and 2^i inclusive. Define the function

$$f_i(p) = F(e)/\beta(e),$$

where e is the interval of order i to which p belongs. Throughout the paper we make the convention that when $\beta(e) = 0$, and consequently $F(e) = 0$, the value 0 shall be assigned to

the meaningless symbol $F(e)/\beta(e)$. Let $h_i(p)$ be, for each p , the upper bound of $f_{i+t}(p)$, and $h(p)$ the limit of the monotone sequence $h_i(p)$. Then $h(p)$ is the upper limit of the sequence $f_i(p)$.

Definition 2. The upper limit $h(p)$ and the corresponding lower limit $g(p)$ are called the upper and lower net-derivatives of $F(e)$ with respect to $\beta(e)$.

Fundamental lemma. If on a set e measurable Borel $h(p) \geq s$, then

$$F(e) \geq s\beta(e).$$

We use the notation $e(f, s)$ to denote the set of points for which $f(p) > s$. $f_i(p)$ is constant over each of the intervals of order i , so that $e_i = e(f_i, s)$ consists of a set of intervals each of which satisfies the inequality above, i.e., $F(e_i) \geq s\beta(e_i)$. Let $e_{i,t} = e_i + e_{i+1} + \dots + e_{i+t-1}$. Then if the inequality is true for $e_{i,t}$ it is true for $e_{i,t+1}$. We may write $e_{i,t+1} = e_{i,t} + e'$, where $e' = e_{i+t}$. $e_{i,t}$ can be divided into a finite number of intervals of order $i+t$, and all of these which belong to e_{i+t} satisfy the inequality. But $e_{i,t}$ and e' have no common point, so that

$$F(e_{i,t+1}) - s\beta(e_{i,t+1}) = F(e_{i,t}) - s\beta(e_{i,t}) + F(e') - s\beta(e') \geq 0.$$

By successive induction the inequality is proved for all $e_{i,t}$. If $E_i = \lim e_{i,t}$ as $t \rightarrow \infty$, since $F(e)$, $\beta(e)$ are additive, $F(E_i) = \lim F(e_{i,t})$, $\beta(E_i) = \lim \beta(e_{i,t})$ and $F(E_i) \geq s\beta(E_i)$. Now $E_i = e(h_i, s)$; for if at a point p , $h_i > s$ then for some t $f_{i+t} > s$ and p belongs to E_i ; while if $h_i(p) \leq s$, $f_{i+t} \leq s$ for all t and p belongs to CE_i .

Let $R(s)$ be the set common to all $E_i(s)$ ($i = 1, 2, \dots$) and $D(s) = \lim R(s - \epsilon)$ as $\epsilon \rightarrow 0$, $\epsilon > 0$; then $D(s)$ is the set of points for which $h(p) \geq s$. For if, at p , $h \geq s$, then for any $\epsilon > 0$, $h_i > s - \epsilon$ for all i ; consequently p belongs to $R(s - \epsilon)$ for any ϵ and therefore to $D(s)$. On the contrary if $h < s$, there is some $\epsilon > 0$ (p fixed) such that $h < s - 2\epsilon$ and a number n can be found so that if $i \geq n$, $h_i < h + \epsilon < s - \epsilon$. This point p does not belong to $R(s - \epsilon)$ and cannot belong to $D(s)$. But

$$\begin{aligned} F[R(s - \epsilon)] &= \lim F[E_i(s - \epsilon)] \text{ as } i \rightarrow \infty, \\ &\geq (s - \epsilon) \lim \beta[E_i(s - \epsilon)] = (s - \epsilon)\beta[R(s - \epsilon)], \\ &\geq s\beta[R(s)] - \epsilon\beta(I), \end{aligned}$$

where I is the fundamental interval, since $\beta(e)$ is non-negative. Then in the limit as $\epsilon \doteq 0$, $F(D) \geq s\beta(D)$. When the original interval is replaced by any sub-interval of the type considered the method of reasoning and the function h are unchanged, so that if e is any such interval or finite sum of such intervals without common points

$$F(eD) \geq s\beta(eD).$$

The inequality will be maintained at successive summations (without common points) and limiting processes. But any set measurable Borel can be obtained from the meshes of the net by these processes, so that if e is measurable Borel, $F(eD) \geq s\beta(eD)$. This is the required lemma with only a difference in expression.

Conclusion. We are now in a position to prove the main theorem. If B is a set contained in D , e any set $eB = eB \cdot D$. In particular the set B where $h(p) < t$, i.e., where the lower limit of $-f_i(p) > -t$, is included in the set where the upper limit of $-f_i(p) \geq -t$. On putting $-F$ in place of F and therefore $-f_i$ in place of f_i , the lemma shows that $-F(eB) \geq -t\beta(eB)$, or $F(eB) \leq t\beta(eB)$. If $A = BD$, i.e., the set where $s \leq h(p) < t$, combining the results,

$$s\beta(eA) \leq F(eA) \leq t\beta(eA).$$

If on a set e , the upper limit of $f_i(p) = h(p) = +\infty$, then $h(p) > s$ for all s and $s\beta(e) \leq F(e)$ so that $\beta(e) = 0$, and consequently $F(e) = 0$. The same can be proved of the set where $h = -\infty$. Thus $h(p)$ is finite everywhere except on a set of β -measure 0. Also from the above inequality and the definition of the generalized Stieltjes integral, $h(p)$ is summable and if E is any set measurable Borel

$$F(E) = \int_E h(p) d\beta(e).$$

By exactly the same reasoning, if $g(p)$ is the lower limit of $f_i(p)$, $g(p)$ is finite "nearly everywhere (β)," is summable and

$$F(E) = \int_E g(p) d\beta(e).$$

The difference of the two integrals is 0, while $h \geq g$ so that $h = g$, i.e., the sequence f_i converges to a single limit $f(p)$

$= h(p) = g(p)$, nearly everywhere (β). This limiting function may be called the net-derivative $D_\beta F$. The main theorem is now proved, since $D_\beta F$ exists and is finite nearly everywhere (β) and if E is measurable Borel,

$$F(E) = \int_E D_\beta F d\beta(e).$$

The proof does not depend on the special method of division; it is sufficient if the "meshes" are divided progressively and if any interval is "Borel measurable" using the meshes as a basic family. The net-derivatives obtained from a finite or countably infinite number of different nets will be the same except on a set of β -measure 0. For if two such net-derivatives are f, k then the sets where $f \geq k$, and where $f < k$ are each measurable Borel and the integral of $f - k$ with respect to β is 0 over each. But the sum of a countably infinite number of sets of zero measure is itself of zero measure. A proof for *all* net-derivatives considered together is lacking but would be desirable. The method of this paper can easily be extended to derivatives in a countably infinite number of dimensions, corresponding to integrals of the same type exhibited by the author.* For this purpose it is convenient, if not necessary, to commence the sub-division in one dimension after another so that a typical mesh of order i would be

$$(m_r - 1)2^{-(i+1-r)} \leq x_r < m_r 2^{-(i+1-r)} \quad (r = 1, 2, \dots, i),$$

$$0 \leq x_r < 1 \quad (r = i+1, i+2, \dots),$$

where m_r is an integer between 1 and 2^{i+1-r} ($r = 1, 2, \dots, i$) inclusive.

RICE INSTITUTE,
HOUSTON, TEXAS.

* P. J. Daniell, *Annals of Mathematics* (1919), vol. 21, p. 30.

SHEFFER'S SET OF FIVE POSTULATES FOR BOOLEAN ALGEBRAS IN TERMS OF THE OPERATION "REJECTION" MADE COM- PLETELY INDEPENDENT.

BY DR. J. S. TAYLOR.

(Read before the American Mathematical Society December 31, 1919.)

Some time ago Professor L. L. Dines* demonstrated the fact that while Sheffer's† five postulates for Boolean algebras in terms of "rejection" are independent in the ordinary sense that no one of the postulates is implied by the other four, they are not completely independent in the sense defined by Professor E. H. Moore‡ inasmuch as the negative of the first postulate implies the third, fourth, and fifth postulates of the set. It is the purpose of this paper to demonstrate that if the first postulate is replaced by one postulating a minimum of four instead of two distinct elements the resulting set of postulates is a completely independent set.§

Sheffer's five postulates concerning a system $\Sigma(K, |)$ are:

1. *There are at least two elements in K .*
2. *Whenever a and b are elements of K , $a | b$ is an element of K .*

Definition: $a' = a | a$.

3. *Whenever a and the indicated combinations of a are elements of K , $(a')' = a$.*

4. *Whenever a , b , and the indicated combinations of a and b are elements of K ,*

$$a | (b | b') = a'.$$

* L. L. Dines. "Complete existential theory of Sheffer's postulates for Boolean algebras," this *BULLETIN*, vol. 21 (Jan., 1915), pp. 183-188.

† H. M. Sheffer, "A set of five postulates for Boolean algebras with applications to logical constants," *Transactions Amer. Math. Society*, vol. 14 (1913), pp. 481-488.

‡ E. H. Moore, "Introduction to a form of general analysis," New Haven Mathematical Colloquium, Yale University Press, page 82.

§ A set of m postulates is said to be ordinarily independent if no one of the m postulates is implied by the others. A set of m postulates is said to be completely independent if, and only if, there are no implicational relations existing among the properties defined either by the postulates as they stand or by the negatives of the postulates. For, if the truth or falsity of one postulate implies either that another postulate is true or that it is false, it would seem either that the two postulates are concerned with two aspects of the same fundamental property, or that there are two fundamental properties involved in such a manner that one of the postulates, at least, deals with both properties.

5. Whenever a, b, c , and the indicated combinations of a, b and c are elements of K ,

$$[a | (b | c)]' = (b' | a) | (c' | a).$$

As stated above, the negative of the first postulate implies the third, fourth, and fifth postulates. For, if K contains less than two distinct elements, these three postulates are satisfied either evidently or vacuously according as the second postulate does or does not hold. This difficulty is no longer encountered, however, if, instead of assuming postulate 1, we assume the following.

1'. There are at least four distinct elements in K .

The truth of this statement is demonstrated by the exhibition of thirty-two systems having the $2^5 = 32$ possible characters $(+++++)$, $(++++-)$, $\dots$, $(-----)$, the i th sign being plus or minus according as the i th postulate is or is not satisfied.

There are two systems with K^{singular} , eight with K^{dual} , six with K^{triple} , and sixteen with $K^{\text{quadruple}}$. In each case the systems have been chosen with as few elements as possible.

In accordance with custom the result of combining elements is given by means of tables. For example, if K contains two elements l_1 and l_2 and if $l_1 | l_1 = l_1$, $l_1 | l_2 = l_1$, $l_2 | l_1 = l_2$, and $l_2 | l_2 = l_2$, this will be stated in the form

$$\begin{array}{c|cc} 1 & l_1 & l_2 \\ \hline l_1 & l_1 & l_1 \\ l_2 & l_2 & l_2 \end{array}$$

If $l_i | l_j$ does not give an element in K , this will be indicated by saying that $l_i | l_j = x$. The extension of this sort of table to systems containing more than two elements is obvious.

The thirty-two systems with their indicated characters follow:

K Singular.

System I₁. $(-++++)$ $l_1 | l_1 = l_1$.

System I₂. $(--++++)$ $l_1 | l_1 \neq l_1$.

K Dual.

System II₁.

$$\begin{array}{c|cc} 1 & l_1 & l_2 \\ \hline (-++++-) & l_1 & l_1 & l_1 \\ & l_2 & l_2 & l_2 \end{array}$$

System II₂.

$$\begin{array}{c|cc} 1 & l_1 & l_2 \\ \hline (-+++ -) & l_1 & l_1 & l_2 \\ & l_2 & l_2 & l_2 \end{array}$$

System II₃.

	1	l_1	l_2
(-+-++)	l_1	l_1	l_1
	l_2	l_1	l_1

System II₄.

	1	l_1	l_2
(-++--)	l_1	l_1	l_2
	l_2	l_1	l_2

System II₅.

	1	l_1	l_2
(-+---)	l_1	l_1	l_2
	l_2	l_2	l_1

System II₆.

	1	l_1	l_2
(--+-+)	l_1	l_1	l_2
	l_2	x	l_2

System II₇.

	1	l_1	l_2
(---++)	l_1	l_1	x
	l_2	x	l_1

System II₈.

	1	l_1	l_2
(-----)	l_1	l_2	l_1
	l_2	x	l_2

*K Triple.*System III₁.

	1	l_1	l_2	l_3
(-+-+-)	l_1	l_1	l_2	l_3
	l_2	l_1	l_1	l_1
	l_3	l_2	l_1	l_2

System III₂.

	1	l_1	l_2	l_3
(--++-)	l_1	l_1	l_2	l_3
	l_2	l_3	l_3	l_1
	l_3	l_2	x	x

System III₃.

	1	l_1	l_2	l_3
(-+----)	l_1	l_1	l_2	l_3
	l_2	l_3	l_1	l_3
	l_3	l_2	l_3	l_2

System III₄.

	1	l_1	l_2	l_3
(--+-)	l_1	l_1	l_2	l_3
	l_2	l_1	l_2	l_3
	l_3	l_1	x	l_3

System III₅.

	1	l_1	l_2	l_3
(----+)	l_1	l_1	l_2	x
	l_2	l_1	l_1	l_1
	l_3	l_2	l_1	l_2

System III₆.

	1	l_1	l_2	l_3
(-----)	l_1	l_1	l_2	x
	l_2	l_3	l_1	l_2
	l_3	x	x	l_2

*K Quadruple.*System IV₁.

	1	l_1	l_2	l_3	l_4
	$\overline{l_1}$	$\overline{l_2}$	$\overline{l_1}$	$\overline{l_4}$	$\overline{l_3}$
(+++++)	l_2	l_1	l_1	l_1	l_1
	l_3	l_4	l_1	l_4	l_1
	l_4	l_3	l_1	l_1	l_3

System IV₂.

	1	l_1	l_2	l_3	l_4
	$\overline{l_1}$	$\overline{l_1}$	$\overline{l_1}$	$\overline{l_1}$	$\overline{l_1}$
(++++-)	l_2	l_2	l_2	l_2	l_2
	l_3	l_3	l_3	l_3	l_3
	l_4	l_4	l_4	l_4	l_4

System IV₃.

	1	l_1	l_2	l_3	l_4
	$\overline{l_1}$	$\overline{l_1}$	$\overline{l_1}$	$\overline{l_1}$	$\overline{l_1}$
(+++ - +)	l_2	l_1	l_2	l_1	l_1
	l_3	l_1	l_1	l_3	l_1
	l_4	l_1	l_1	l_1	l_4

System IV₄.

	1	l_1	l_2	l_3	l_4
	$\overline{l_1}$	$\overline{l_1}$	$\overline{l_1}$	$\overline{l_1}$	$\overline{l_1}$
(++ - + +)	l_2	l_1	l_1	l_1	l_1
	l_3	l_1	l_1	l_1	l_1
	l_4	l_1	l_1	l_1	l_1

System IV₅.

	1	l_1	l_2	l_3	l_4
	$\overline{l_1}$	x	x	x	x
(+ - + + +)	l_2	x	x	x	x
	l_3	x	x	x	x
	l_4	x	x	x	x

System IV₆.

	1	l_1	l_2	l_3	l_4
	$\overline{l_1}$	$\overline{l_1}$	l_2	l_3	l_4
(+++ - -)	l_2	l_1	l_2	l_3	l_4
	l_3	l_1	l_2	l_3	l_4
	l_4	l_1	l_2	l_3	l_4

System IV₇.

	1	l_1	l_2	l_3	l_4
	$\overline{l_1}$	l_2	l_2	l_1	l_1
(++ - + -)	l_2	l_1	l_1	l_1	l_1
	l_3	l_1	l_1	l_1	l_1
	l_4	l_1	l_1	l_1	l_1

System IV₈.

	1	l_1	l_2	l_3	l_4
	$\overline{l_1}$	$\overline{l_1}$	l_1	x	x
(+ - + + -)	l_2	l_2	l_2	x	x
	l_3	x	x	x	x
	l_4	x	x	x	x

System IV₉.

	1	l_1	l_2	l_3	l_4
	$\overline{l_1}$	$\overline{l_1}$	l_2	l_1	l_1
(++ - - +)	l_2	l_2	l_1	l_1	l_1
	l_3	l_1	l_1	l_1	l_1
	l_4	l_1	l_1	l_1	l_1

System IV₁₀.

	1	l_1	l_2	l_3	l_4
	$\overline{l_1}$	$\overline{l_1}$	l_2	x	x
(+ - + - +)	l_2	x	l_2	x	x
	l_3	x	x	x	x
	l_4	x	x	x	x

System IV₁₁.

	1	l_1	l_2	l_3	l_4
	l_1	l_1	x	x	x
(+ - - + +)	l_2	x	l_1	x	x
	l_3	x	x	x	x
	l_4	x	x	x	x

System IV₁₂.

	1	l_1	l_2	l_3	l_4
	l_1	l_2	l_1	l_1	l_1
(+ + - - -)	l_2	l_1	l_3	l_1	l_1
	l_3	l_3	l_3	l_3	l_3
	l_4	l_4	l_4	l_4	l_4

System IV₁₃.

	1	l_1	l_2	l_3	l_4
	l_1	l_1	l_2	x	x
(+ - + - -)	l_2	l_1	l_2	x	x
	l_3	x	x	x	x
	l_4	x	x	x	x

System IV₁₄.

	1	l_1	l_2	l_3	l_4
	l_1	l_1	l_2	l_3	x
(+ - - + -)	l_2	l_1	l_1	l_1	x
	l_3	l_2	l_1	l_2	x
	l_4	x	x	x	x

System IV₁₅.

	1	l_1	l_2	l_3	l_4
	l_1	l_2	l_1	x	x
(+ - - - +)	l_2	x	l_2	x	x
	l_3	x	x	x	x
	l_4	x	x	x	x

System IV₁₆.

	1	l_1	l_2	l_3	l_4
	l_1	l_1	l_2	x	x
(+ - - - -)	l_2	l_3	l_1	l_2	x
	l_3	x	x	l_2	x
	l_4	x	x	x	x

It may be of interest to note that a similar change in the first postulate of Bernstein's* set of four postulates in terms of the operator "rejection" also makes that set completely independent, as I have shown in an earlier paper,† and that furthermore this same change makes my own set of five postulates in terms of "exception"‡ completely independent (together with a change in the fifth postulate at a sacrifice of simplicity). It would be interesting to ascertain whether this postulation of a minimum of four distinct elements is sufficient to being about the complete independence of any

* B. A. Bernstein, "A set of four independent postulates for Boolean algebras," *Transactions Amer. Math. Society*, vol. 17 (1916), pp. 50-52.

† J. S. Taylor, "Complete existential theory of Bernstein's set of four postulates for Boolean algebras," *Annals of Mathematics*, second series, vol. 19, no. 1 (Sept., 1917), pp. 64-69.

‡ J. S. Taylor, "A set of five postulates for Boolean algebras in terms of the operation 'exception,'" *University of California Publications in Mathematics*, vol. 1 (April 12, 1920), pp. 241-248.

set of postulates for Boolean algebras where the remaining postulates of the set are already free from all implicational relationships among themselves.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

ROTATION SURFACES OF CONSTANT CURVATURE IN SPACE OF FOUR DIMENSIONS.

BY PROFESSOR C. L. E. MOORE.

(Read before the American Mathematical Society April 24, 1920.)

1. IN space of four dimensions there are two special rotations each of which has circles for path curves. In this note I shall discuss the surfaces generated by these special rotations which have constant curvature. The first type is given by the equations

$$(1) \quad \begin{aligned} X &= x \cos t - y \sin t, & Y &= x \sin t + y \cos t, \\ Z &= z, & W &= w. \end{aligned}$$

This rotation leaves each point of the zw -plane invariant and any plane completely perpendicular to it is left invariant as a plane but not point for point. The rotation then is simply isomorphic with a rotation in the xy -plane.*

If the curve

$$(c) \quad x = x(s), \quad y = y(s), \quad z = z(s), \quad w = w(s),$$

where s denotes arc length measured from a fixed point, is rotated, equations (1) are the parametric equations of the surface generated. The parameter curves $s = \text{const.}$, $t = \text{const.}$ will be orthogonal if

$$(2) \quad xy' - x'y = 0 \quad \text{or} \quad y = kx,$$

where primes denote derivatives with respect to s . Hence the meridian curves (orthogonal trajectories of the path curves) on a surface generated by rotation (1), lie in a 3-space which contains the absolutely invariant plane. If the meridian curve

* Phillips and Moore, "Rotations in space of even dimensions," *Proceedings Amer. Academy*, vol. 55.

is a plane curve its plane must lie in one of these 3-spaces and consequently cut the xw -plane in a line. It will then also cut a set of invariant planes which are completely perpendicular to the xw -plane in lines. A plane which cuts the xw -plane in a line will then generate a 3-space by the rotation (1). Hence the surfaces of rotation, generated by (1), for which the meridians are plane curves must lie in a 3-space. Such surfaces of constant curvature are already well known.

If the meridians are not plane curves they must satisfy (2), and for simplicity we will assume that it lies in the xzw -space. The element of arc on such a surface is

$$(3) \quad d\sigma^2 = ds^2 + x^2 dt^2,$$

where s denotes the arc on the meridian curve and σ the arc of any curve traced on the surface of rotation. If the surface has constant positive curvature,*

$$\frac{1}{a^2} = -\frac{1}{x} \frac{d^2 x}{dt^2}.$$

The integral of this is

$$(4) \quad x = c \cos (s/a + b).$$

By properly choosing the reference system b can be made to vanish. This we will assume done. We can then write

$$(5) \quad \begin{aligned} dx^2 + dz^2 + dw^2 &= ds^2 \quad \text{or} \\ dz^2 + dw^2 &= \left(1 - \frac{c^2}{a^2} \sin^2 \frac{s}{a}\right) ds^2. \end{aligned}$$

For a rotation surface in 3-space equation (5) reduces to

$$(6) \quad dz^2 = \left(1 - \frac{c^2}{a^2} \sin^2 \frac{s}{a}\right) ds^2,$$

the solution of which is well known. For the case here considered, $dz^2 + dw^2 = ds^2$ is the square of the element of arc of the projection of the meridian curve on the xw -plane. Equation (6) then becomes

$$ds_1^2 = \left(1 - \frac{c^2}{a^2} \sin^2 \frac{s}{a}\right) ds^2.$$

* Eisenhart, *Differential Geometry*, p. 270.

The solution for s_1 in terms of s is the same as the solution of (6) for z in terms of s . The projection s_1 then bears the same relation to x that z does for a rotation surface of constant positive curvature in 3-space. Hence to obtain curves in the xzw -space which can be used as meridian curves of a surface of revolution of constant positive curvature, trace on the xw -plane curves (c) which will generate such surfaces. Then roll this plane into a cylinder with elements parallel to OX , making OZ roll into the section of the cylinder in the xw -plane. The curves into which the curves (c) roll are the curves sought.

2. The second kind of rotation which has circular path curves is

$$(7) \quad \begin{aligned} X &= x \cos t - y \sin t, & Y &= x \sin t + y \cos t, \\ Z &= z \cos t - w \sin t, & W &= z \sin t + w \cos t. \end{aligned}$$

This rotation leaves ∞^2 planes of a linear congruence invariant.* These can be arranged into pairs of completely perpendicular planes. If the curve (c) is rotated (7) will be the parametric equations of the resulting surface. The parameter curves on this surface will be orthogonal if

$$(8) \quad x'y - xy' + z'w - zw' = 0.$$

Now by this rotation† if a line cuts one path curve orthogonally it will cut all orthogonally and consequently must lie in a plane which cuts two completely perpendicular invariant planes in lines. Therefore the solutions of (8) must lie on cones whose tangent planes cut a pair of completely perpendicular invariant planes in lines. This equation shows that no 3-space curve lying in a 3-space passing through the origin can be used as a meridian curve. For by choosing the axes properly such a 3-space can be represented by $x = 0$ and consequently, from (8), $z = kw$ and hence the curve must be plane. If the curve is plane it must lie in a plane cutting two completely perpendicular invariant planes in lines. We will discuss this case first.

Let the curve lie in the xz -plane. Equations (7) then become

$$X = x \cos t, \quad Y = x \sin t, \quad Z = z \cos t, \quad W = z \sin t.$$

* Moore, "Rotations in hyperspace," *Proceedings Amer. Academy*, vol. 53.

† Moore, "Surfaces of rotation in space of four dimensions," *Annals of Math.*, vol. 21.

The element of arc of the surface formed by rotating $x = x(s)$, $z = z(s)$ is

$$(9) \quad d\sigma^2 = ds^2 + (x^2 + z^2)dt^2.$$

If the surface has constant positive curvature, then

$$(10) \quad \rho = \sqrt{x^2 + z^2} = c \cos \frac{s}{a}.$$

Using polar coordinates in the xz -plane, we then have

$$(11) \quad d\rho^2 + \rho^2 d\theta^2 = \frac{a^2 d\rho^2}{c^2 - \rho^2} \quad \text{or} \quad \frac{1}{\rho} \sqrt{\frac{\rho^2 + a^2 - c^2}{c^2 - \rho^2}} d\rho = d\theta.$$

Just as in three dimensions, there are three types of curves depending on the relations between a and c .

(a) If $a = c$, the solution is

$$\rho = a \sin (\theta + b).$$

Thus any circle of radius $a/2$ passing through the origin will rotate, by (7), into a surface of constant positive curvature a . If the circle has its center at the origin, it will generate a developable surface as is at once evident from (9).

(b) If $a < c$, the solution is

$$\begin{aligned} & \frac{1}{2} \sin^{-1} \frac{2\rho^2 + a^2 - 2c^2}{a^2} \\ & - \frac{1}{2} \sqrt{\frac{c^2 - a^2}{c^2}} \sin^{-1} \frac{2c^2(a^2 - c^2) + (2c^2 - a^2)\rho^2}{c^2\rho^2} = \theta + b. \end{aligned}$$

(c) If $a > c$, the solution is

$$\begin{aligned} & \frac{1}{2} \sin^{-1} \frac{2\rho^2 + a^2 - 2c^2}{a^2} \\ & - \frac{1}{2} \sqrt{\frac{a^2 - c^2}{c^2}} \cosh^{-1} \frac{2c^2(a^2 - c^2) + (2c^2 - a^2)\rho^2}{c^2\rho^2} = \theta + b. \end{aligned}$$

In cases (a) and (b) the solutions can be written in finite form, while the corresponding solutions for 3-space can be written in parametric form only by the use of elliptic functions.*

* See Bianchi, vol. 1, p. 233.

To determine the form of these curves we return to equation (11) and write it in the form

$$\sqrt{\frac{\rho^2 + a^2 - c^2}{c^2 - \rho^2}} = \frac{\rho d\theta}{d\rho} = \tan \psi,$$

where ψ is the angle between the tangent line and the radius vector.

If $a < c$, this last equation shows that $c^2 - a^2 < \rho^2 < c^2$ and hence the curve lies between two concentric circles of radii $\sqrt{c^2 - a^2}$ and c . Also at the inner circle ψ is zero and at the outer circle $\pi/2$. This curve then consists of segments which start perpendicular to the inner circle, touch the outer circle and return perpendicular to the inner circle. If $a > c$ the curve is in the form of a loop which crosses at the origin and is tangent to the circle $\rho = c$. For given values of a and c equation (11) shows that all curves cut the circle $\rho = \text{const.}$ at the same angle and have the same radius of curvature at the points of intersection. The curves then are symmetrical about the line joining the origin to the point of tangency with $\rho = c$. All curves are obtained by rotating a given one about the origin.

3. If the surface is one of constant negative curvature,

$$\frac{1}{a^2} = \frac{1}{\rho} \frac{d^2\rho}{dt^2}.$$

Hence $\rho = c_1 \cosh s/a + c_2 \sinh s/a$.

The three cases to consider are $c_1 = 0$; $c_2 = 0$; $c_1 = c_2$. The square of the element of arc in each of these three cases is

$$(i) \quad d\sigma^2 = ds^2 + c^2 \cosh^2 s/a \, dt^2,$$

$$(ii) \quad d\sigma^2 = ds^2 + c^2 \sinh^2 s/a \, dt^2,$$

$$(iii) \quad d\sigma^2 = ds^2 + c^2 e^{2s/a} \, dt^2.$$

Any case other than these may be obtained by taking for ρ either of the values $\cosh (s/a + b)$, $\sinh (s/a + b)$ where b is constant. For the first case, following the previous discussion, we have

$$(12) \quad d\rho^2 + \rho^2 d\theta^2 = \frac{a^2 d\rho^2}{\rho^2 - c^2} \quad \text{or} \quad \sqrt{\frac{\rho^2 - c^2 - a^2}{c^2 - \rho^2}} = \frac{\rho d\theta}{d\rho}.$$

In this case also the equation is readily integrated in terms

of elementary functions, but we will obtain the properties of the curve from equation (12) itself. The first thing observed is that $c^2 < \rho^2 < c^2 + a^2$. Hence the curve does not pass through the origin for any value of c but lies between two concentric circles with center at the origin as before. In this case however the curve is tangent to the inner circle and perpendicular to the outer circle. This corresponds to the pseudospherical surface of hyperbolic type.

In case (ii) the differential equation is

$$\sqrt{\frac{a^2 - c^2 - \rho^2}{c^2 + \rho^2}} = \frac{\rho d\theta}{d\rho}.$$

In this case, then, $a > c$ and $\rho^2 < a^2 - c^2$. The curve then lies inside the circle $\rho = \sqrt{a^2 - c^2}$ to which it is perpendicular. As the value of ρ decreases, the value of ψ approaches $\tan^{-1} \sqrt{(a^2 - c^2)/a^2}$ and therefore the curve approaches the origin, cutting all the radius vectors at the above angle.

In case (iii) we have

$$\frac{1}{\rho} \sqrt{a^2 - \rho^2} = \frac{\rho d\theta}{d\rho}.$$

The curve then lies inside the circle $\rho = a$ which it cuts perpendicularly. As ρ decreases, ψ approaches $\pi/2$. This curve then approaches the origin, cutting the radius vectors at right angles. This is the limiting case of either of the above. If any of the above curves are rotated about the origin in the xz -plane, when rotated by (7) they will still generate a surface of constant negative curvature. The forms of all these differentials lend themselves readily to graphical solution so that a picture of the curves is easily obtained.

4. In case of the general rotation (7) the square of the element of arc is

$$(13) \quad d\sigma^2 = ds^2 + (x^2 + y^2 + z^2 + w^2)dt^2$$

and if the surface is one of constant positive curvature,

$$(14) \quad \rho = \sqrt{x^2 + y^2 + z^2 + w^2} = c \cos \frac{s}{a}.$$

Consequently the distance ρ of the points from the origin is a function of s . Then if we have a curve (c) satisfying

equation (14) and unroll the cone which projects this curve from the origin, on the xz -plane, the curve into which (c) goes will satisfy (10) and hence will rotate into a surface of constant positive curvature. Hence *the curves of 4-space which rotate by (7) into surfaces of constant positive curvature are obtained by tracing on the xz -plane curves (c) which rotate by (7) into surfaces of constant positive curvature, and then rolling the xz -plane into a cone with vertex at the origin. The curves into which the curves (c) go are those sought.* Of course to obtain curves which cut the path curves orthogonally the above cones must be such that each of their tangent planes cuts a pair of completely perpendicular planes in lines. Curves which generate surfaces of constant negative curvature are obtained in a similar manner.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

SHORTER NOTICES.

The Mystery of Space. A study of the hyperspace movement in the light of the evolution of new psychic faculties, and an inquiry into the genesis and essential nature of space. By ROBERT T. BROWNE. New York, E. P. Dutton and Company, 1919. 8vo. 395 pp. \$4.00.

THIS book is a mixture of cosmogony, psychology, and geometry; heralded in a recent flyer as "an epoch-making work."

With the cosmogony and psychology we can have little to do. Anyone obviously has a perfect right to philosophize about the universe and the true nature of space as much as he pleases and to dress his philosophy in Greek nomenclature to give it a scientific aspect if he chooses. But the reader may be pardoned if he is tempted to compare the periodic wanderings of Mr. Browne's "monopyknon," from chaos through seven stages of "pyknosis" before emerging into physical being, and thence through sentient, mental, and spiritual stages back to chaos again, with Goethe's story of the Homunculus; and to assert that the one is as mediaeval in character as the other was intended to be. Nor can one deny to the author the right to hold and to defend, if he can, the theory that humanity will one day develop to the point where,

through the agency of the pineal gland and the pituitary body, intuition will supersede intellect and mathematics will cease to be useful or, if needed, will be known intuitively to the infants of that millennium. If a common mortal finds difficulty in following the arguments in support of this theory, it is probably because he has regarded his pineal gland and pituitary body, if conscious of them at all, as rudimentary rather than embryonic, and so has failed to cultivate their latent powers, as Mr. Browne would have us believe clairvoyants, and their like, do.

But with geometry the case is somewhat different. The author's profound apology to mathematicians for his temerity in entering this field indicates that he was conscious of his lack of training for the work he essayed to do. That he attempted to prepare for it by reading about non-euclidean geometry and hyperspace, is shown by a more or less detailed reproduction of the history of non-euclidean geometry from the earliest attempts to prove Euclid's parallel postulate to the present time; by numerous quotations from various writers on non-euclidean geometry and hyperspace; and by an appendix containing a bibliography of over one hundred articles and books. That he failed to digest this material is shown by the persistent confusion of non-euclidean geometry with hyperspace, and by his supposition that these subjects were invented by mathematicians for the purpose of explaining the real nature of space. Of course he must refute these spurious explanations in order to introduce his own idea of space. Naturally he can offer no proof that real space is not non-euclidean in character or that it has exactly three dimensions, for he scorns basic axioms and takes mathematicians to task for deifying the definition. One wonders why he did not seize upon hyperbolic geometry as a coup de maître in support of his theory that real space is finite, limited, and surrounded by chaos, or "spacelessness." The best he can do, however, is to assert the absurdity of all geometrical conceptions, other than the classical euclidean, styling them "superfoetated hypotheses" or "mathetic divertissements." It is apparently an easy and interesting game for Mr. Browne to tilt rhetorical lances at non-euclidean geometry and the fourth dimension and he indulges frequently in this sport throughout the book. But this is a dangerous game for one who apologizes upon entering it and exhibits throughout a lack of knowledge of

the objects at which he aims his lances, and our author must not expect to fare better at it than did his prototype who mistook the windmills for something other than they were.

L. WAYLAND DOWLING.

Junior High School Mathematics, Books 1 and 2. By E. H. TAYLOR and FISKE ALLEN. New York, Henry Holt and Company, 1919.

To one who has followed the trend of mathematical education during the last ten or fifteen years, it is apparent that the voluminous discussion on coordination of elementary mathematics is beginning to bear fruit. Recent elementary texts like the one under discussion show very clearly that teachers and authors now have a definite aim in view. The two most notable features of these modern texts are, first, the early introduction of geometric ideas and constructions without formal demonstrations, and second, the socializing of elementary mathematics by interesting the pupil in everyday activities which require computations and geometric constructions.

The text under review is a modern coordinated treatment of arithmetic, algebra and geometry for use in the seventh and eighth grades. Book 1 contains first a review of the fundamental operations of arithmetic with numerous problems in keeping accounts and simple business transactions. This is followed by a study of the formula as an introduction to literal arithmetic and algebra; percentage and its simpler practical applications; measurement of lines and angles; triangles and other constructions with ruler and compasses; parallel lines, quadrilaterals and polygons; and the mensuration of areas. The arrangement of subject matter is excellent, and the problem material covers a very extensive range of useful applications.

Book 2 continues the three lines of work begun in the first volume, viz., arithmetic, algebra and geometry. Algebra is approached through the formula and the use of letters to shorten statements in words; the square root of numbers is found arithmetically and also graphically by using the Pythagorean theorem; ratio and proportion are also treated both arithmetically and geometrically; this is followed by the mensuration of simple solids with numerous concrete problems; explanation of the negative; problems involving simple

equations; graphs and their application to statistical data; and the technical applications of percentage, such as banking and investment.

The book closes with a few extended problems which follow some special activity involving mathematical calculations through all its branches. A typical problem is the farm project, which includes financing the purchase and equipment of the farm; the construction of farm buildings; the purchase of live stock and implements; various agricultural problems; and finally the calculation of expenses and returns on the investment. Such problem material provides the most effective form of coordination and also instills valuable lessons in thrift and industry, so much needed at the present time.

Such books as these mark a distinct advance in the teaching of elementary mathematics, and cannot fail to make teaching more efficient and lead to a better appreciation of what mathematics is, and of its importance as a factor in common school education.

S. E. SLOCUM.

Tables from the Mathematical Theory of Investment. By E. B. SKINNER. Boston, Ginn and Company, 1917. 26 pages. Price 36 cents.

Problems in the Mathematical Theory of Investment. By G. R. CLEMENTS. Boston, Ginn and Company, 1917. 24 pages. Price 32 cents.

THESE tables are reprinted without change or introductory explanation from Professor Skinner's *Mathematical Theory of Investment*. In most cases, however, the descriptive titles of the tables are sufficient to make the use of them clear to persons who have not read the book from which they are reprinted.

Professor Skinner's textbook, first published in 1913, has been widely adopted and is generally recognized as the best American textbook on the subject. In class exercises and in final examinations it is frequently convenient to have the students use the tables without using the text. The printing of the tables separately was therefore highly desirable.

The collection of problems by G. R. Clements contains one hundred examples of the same general nature as those in Professor Skinner's text. The problems are not, however,

graded by subjects, so that the student is required to determine for himself the class into which each problem falls and to select the formula required for its solution. The problems are preceded by three pages of hints as to how to analyze and solve a problem. The more important formulas used are listed. Hints are given in connection with some of the more difficult examples. A complete set of answers is printed at the end of the book.

This collection will be of great assistance to teachers who wish a supplementary list of miscellaneous examples for review; or who wish to use the textbook for a series of years without the necessity of assigning the same problems year after year.

CLINTON H. CURRIER.

NOTES.

PROFESSORS L. E. DICKSON, of the University of Chicago, and L. P. EISENHART, of Princeton University, have been elected delegates of the American section of the International mathematical union to attend the meeting of the union at the University of Strasbourg beginning September 18, 1920.

THE April number (volume 42, number 2) of the *American Journal of Mathematics* contains the following papers: "On the convergence of certain classes of series of functions," by R. D. CARMICHAEL; "On the solution of linear equations in infinitely many variables by successive approximations," by J. L. WALSH; "Self-dual plane curves of the fourth order," by L. E. WEAR; "On the groups of isomorphisms of a system of abelian groups of order p^m and type $(n, 1, 1, \dots, 1)$," by L. C. MATHEWSON; "On the satellite line of the cubic," by R. M. WINGER.

AT the meeting of the Edinburgh mathematical society on May 14, the following papers were read: By E. T. WHITTAKER, "Some developments in curve fitting and the calculus of differences"; by C. G. KNOTT, "Robert Hooke on molecular interplay." At the meeting of June 11, a paper was read by

G. SMEAL, "On direct and inverse interpolation by divided differences."

BEGINNING with volume 81, the *Mathematische Annalen* will be published by Julius Springer, of Berlin, instead of by Teubner as heretofore.

THE following university courses in mathematics are announced for the academic year 1920-1921:

PRINCETON UNIVERSITY. By Professor H. B. FINE: Theory of functions, three hours.—By Professors L. P. EISENHART and OSWALD VEULEN: Seminar in relativity, three hours.—By Professor OSWALD VEULEN: Projective geometry, three hours.—By Professor J. H. M. WEDDERBURN: Theory of functions of a real variable, three hours.—By Professor J. W. ALEXANDER: Conformal representation, three hours.

STANFORD UNIVERSITY. First quarter (October-December):—By Professor R. E. ALLARDICE: Advanced calculus, five hours.—By Professor H. F. BLICHFELDT: Vector analysis, four hours. Second quarter (January-March):—By Professor GREEN: Modern coordinate geometry, five hours.—By Professor R. E. ALLARDICE: Projective geometry, five hours.—By Professor H. F. BLICHFELDT: Differential equations, five hours. Third quarter (April-June):—By Professor GREEN: Modern coordinate geometry, five hours.—By Professor R. E. ALLARDICE: Theory of functions, five hours.—By Professor H. F. BLICHFELDT: Non-euclidean geometry, four hours.

THE Buchhandlung Gustav Fock, of Leipzig, offers for sale the library of the late Professor MORITZ CANTOR, of Heidelberg, the historian of mathematics. The library consists of about 2,000 volumes and 2,500 pamphlets.

At the University of Berlin, Professor L. E. BROUWER, of the University of Amsterdam, has been appointed professor of mathematics, Professor R. VON MISES, of the Dresden technical school, has been appointed professor of applied mathematics, and Dr. ISSAI SCHUR has been promoted to a full professorship of mathematics. Professor C. CARATHÉODORY has resigned, to accept a professorship at the National University at Athens.

At the University of Clermont, Dr. G. GIRAUD has been appointed chargé de cours for the differential and integral calculus, in place of Professor A. C. E. PELLET, who has retired from active teaching.

DR. RENÉ GARNIER has been appointed chargé de cours for theoretical and applied mechanics at the University of Poitiers.

At the University of Nancy, Dr. LÉOPOLD LEAU has been appointed professor of the differential and integral calculus, as successor to Dr. A. S. E. HUSSON.

DR. L. SIRE, of the University of Rennes, has been appointed professor of applied mathematics at the University of Lyons, as successor to the late Professor D. J. B. FLAMME.

PROFESSOR H. LAMB, of the University of Manchester, Sir T. L. HEATH, and Professor W. H. BRAGG, of the University of London, have been elected honorary fellows of Trinity College, Cambridge.

THE senate of the University of Dublin has conferred the honorary degree of doctor of science on Professor W. H. BRAGG, of the University of London, and Professor R. A. MILLIKAN, of the University of Chicago.

MR. W. E. H. BERWICK has been appointed lecturer in mathematics at the University of Leeds.

PROFESSOR JACQUES HADAMARD has been elected a foreign honorary member of the American academy of arts and sciences.

PROFESSOR AUGUSTUS TROWBRIDGE, of the department of physics of Princeton University, has been appointed chairman of the division of astronomy, mathematics and physics of the National research council, for the year beginning July 1, 1920.

AMONG the grants for research made by the American association for the advancement of science in April are three hundred dollars to Professor S. LEFSCHETZ, of the University

of Kansas, to assist in the publication of his memoir on algebraic surfaces, which was awarded the Bordin prize by the Paris academy of sciences, and one hundred dollars to Professor OLIVE C. HAZLETT, of Mount Holyoke College, in support of her work in the theory of hypercomplex numbers and invariants.

ON the recommendation of the National academy of sciences, Columbia University has awarded its Barnard medal for meritorious service to science to Professor ALBERT EINSTEIN, for "his highly original and fruitful development of the fundamental concepts of physics through the application of mathematics."

DR. R. S. WOODWARD will retire from the presidency of the Carnegie Institution of Washington at the end of the present year, after sixteen years of service.

DR. J. W. ALEXANDER has been appointed assistant professor of mathematics at Princeton University.

IN the department of mathematics at the University of Minnesota, associate professor W. H. BUSSEY has been promoted to a full professorship and appointed assistant dean of the college of science, literature and the arts, and assistant professors A. L. UNDERHILL, R. W. BRINK, and W. L. HART have been promoted to associate professorships. Mr. F. K. BAIER, JR., of Pennsylvania State College, and Dr. GLADYS GIBBENS, of the University of Chicago, have been appointed instructors.

ASSOCIATE professor J. E. ROWE, of the Pennsylvania State College, was promoted to a full professorship of mathematics in January, 1920.

ASSISTANT professor C. W. WESTER, of Iowa State Teachers' College, has been promoted to a full professorship of mathematics.

AT Oberlin College, associate professor W. D. CAIRNS has been made full professor and head of the department of mathematics. Professor FREDERICK ANDEREGG has retired from active teaching, after thirty-three years of service.

DR. J. R. MUSSELMAN, of Washington University, has been appointed associate in mathematics at Johns Hopkins University.

MR. JESSE DOUGLAS has been appointed instructor in mathematics at Columbia University.

At the Carnegie Institute of Technology, assistant professor J. R. EVERETT, of Baker University, and Professor G. W. HESS, of Bethany College, have been appointed instructors in mathematics.

ASSISTANT professor A. E. BABBITT, of the University of Nebraska, has resigned to enter actuarial work.

MR. C. GOUWNES, of the University of Kansas, has been appointed assistant professor of mathematics at the State College of Iowa.

DR. ANNA M. HOWE has been appointed instructor in mathematics at the Sophie Newcomb College of Tulane University.

MR. H. B. MEEK has been appointed instructor in mathematics at Yale University.

MISS MARIAN M. TORREY has been appointed instructor in mathematics at the University of West Virginia.

MISS MAY J. SPERRY, of Brown University, has been appointed instructor in mathematics and physics at Knox College, Galesburg, Ill.

THE death is reported of the Indian mathematician, SRINIVASA RAMANUJAN, F. R. S., fellow of Trinity College, Cambridge, at the age of thirty-two years.

PROFESSOR P. VAN GEER died at the Hague, on October 3, 1919, at the age of seventy-eight years.

PROFESSOR M. HAID, of the Karlsruhe technical school, died November 15, 1919.

THE death is reported of Professor A. BOERSCH, at Homburg.

THE death is reported of Professor R. HEGER, at Dresden, at the age of seventy-three years.

PROFESSOR PAUL STÄCKEL, of the University of Heidelberg, died December 13, 1919, at the age of fifty-seven years.

THE death is reported of Professor R. MALSTROEM, of the department of mechanics at the technical school at Helsingfors.

BOOK CATALOGUES: Buchhandlung Gustav Fock, Leipzig, Handapparate und Sammlungen von Dissertationen und Programmen auf dem Gebiete der Naturwissenschaften sowie der Mathematik.—Gauthier-Villars, Paris, Bulletin des Publications nouvelles, 4e trimestre, 1919.—Rudolf Geering, Basel, Antiquariats-Katalog Nr. 383 (containing the library of the late Professor J. H. Graf, of Bern).—B. G. Teubner, Leipzig, Neuerscheinungen und Neuauflagen, 1914–1919.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

CHILD (J. M.). The early mathematical manuscripts of Leibniz. Translated from the Latin texts published by Carl Immanuel Gerhardt with critical and historical notes by J. M. Child. Chicago and London, Open Court, 1920. 8vo. 4 + 238 pp.

EUCLID. See HEATH (T. L.).

GERHARDT (C. I.). See CHILD (J. M.).

GROSSE (W.). Graphische Papiere und ihre vielseitige Anwendung. Düren, Schleicher und Schüll, 1919. 179 pp.

HEATH (T. L.). Euclid in Greek. Book I, with introduction and notes. Cambridge, University Press, 1920. 12mo. 240 pp. 10s.

LABOUREUR (M.). Cours d'exercices sur le calcul mathématique, algèbre, différentiel et intégral. Paris et Liège, Béranger, 1920. 8vo. 8 + 318 pp. Fr. 30.00

LEIBNIZ (G. W.). See CHILD (J. M.).

II. ELEMENTARY MATHEMATICS.

AMMERMAN (C.). See FORD (W. B.).

BALL (K. F.) and WEST (M. E.). Household arithmetic. (Lippincott's unit texts.) Philadelphia, Lippincott, 1920. 8vo. 271 pp.

- BEHRENDSEN (O.) und GÖTTING (E.). Lehrbuch der Mathematik nach modernen Grundsätzen. Oberstufe. Ausgabe A. 2te Auflage. Leipzig, Teubner, 1919. 265 pp. M. 5.00
- BETTINI (B.) e CIAMBERLINI (C.). Elementi di aritmetica e geometria, per la 2a classe ginnasiale. 3a edizione riveduta e migliorata. Firenze, R. Bemporad e figlio (M. Ricci), 1919. 16mo. 156 pp. L. 3.00
- BOWYER (L. H.) and MCINTOSH (C.). Eighth year arithmetic adapted to the Illinois state course of study. Sixth general revision. Monticello, Ill., McIntosh, 1919. 8vo. 151 pp. \$0.60
- CIAMBERLINI (C.). See BETTINI (B.).
- CLARK (J. R.). See RUGG (H. O.).
- FORD (W. B.) and AMMERMAN (C.). Second course in algebra. New York, Macmillan, 1920. 9 + 299 pp. \$1.28
- GAZZANIGA (P.). See VERONESE (G.).
- GÖTTING (E.). See BEHRENDSEN (O.).
- HART (C.) and WATTS (M. J.). Commercial and industrial arithmetic for students of high school grade. New York, Appleton, 1919. 8vo. 6 + 438 pp. \$2.00
- MCINTOSH (C.). See BOWYER (L. H.).
- RUGG (H. O.) and CLARK (J. R.). Fundamentals of high school mathematics; a text-book designed to follow arithmetic. Yonkers-on-Hudson, World Book Company, 1919. 12mo. 15 + 368 pp. \$1.60
- SCHWERING (K.). Sammlung von Aufgaben aus der Arithmetik für höhere Lehranstalten. 1ter Lehrgang. 4te Auflage. Freiburg, Herder, 1918. 63 pp. Geb. M. 2.00
- SPERA (S.). Elementi di algebra, per le scuole tecniche e medie inferiori. 2a edizione, riveduta e corretta. Torino, ditta G. B. Paravia e C. (Milano, stamp. ed. Lombarda, di L. Mondaini), 1919. 16mo. 119 pp. L. 3.00
- TESTI (G. M.). Corso di aritmetica con 500 esercizi, ad uso degli alunni delle scuole tecniche, dei ginnasi inferiori e delle scuole complementari. 22a edizione riveduta. Livorno, Giusti, 1920. 16mo. 14 + 307 pp. L. 4.40
- . Corso di matematiche ad uso delle scuole secondarie superiori e più specialmente degli istituti tecnici. Volume 2: Algebra elementare. 9a edizione riveduta. Livorno, Giusti, 1920. 8vo. 11 + 431 pp. L. 7.60
- VERONESE (G.). Nozioni elementari di geometria intuitiva ad uso dei ginnasi inferiori. 5a edizione. Padova, fratelli Drucker (Società coop. tipografica), 1919. 8vo. 100 pp.
- VERONESE (G.) e GAZZANIGA (P.). Elementi di geometria intuitiva, ad uso delle scuole tecniche e complementari. 4a edizione. Padova, fratelli Drucker (Società coop. tipografica), 1919. 8vo. 8 + 163 pp. L. 3.00
- WATTS (M. J.). See HART (C.).
- WEBBER (W. B.). Elementary applied mathematics. A practical course for general students. New York, Wiley, 1920. 12mo. 10 + 115 pp. \$1.25
- WEST (M. E.). See BALL (K. F.).

III. APPLIED MATHEMATICS.

- AMALDI (U.). *Meccanica razionale: lezioni tenute al corso d'integrazione, aprile-luglio 1919, nella r. scuole d'applicazione per gli ingegneri di Padova.* Padova, Parisotto, 1919. 8vo. 411 pp.
- ANDREWS (S. T. G.) and BENSON (S. F.). *The theory and practice of aeroplane design.* London, Chapman and Hall, 1920. 8vo. 454 pp. 15s. 6d.
- BARKER, (E. H.). *Applied mathematics for high school.* Boston, Allyn and Bacon, 1919.
- BENSON (S. F.). See ANDREWS (S. T. G.).
- BERGET (A.). *Topographie.* Paris, Larousse, 1920. 8vo. 328 pp. Fr. 12.00
- CHINI (M.). *Corso speciale di matematiche con numerose applicazioni ad uso dei chimici e dei naturalisti. 4a edizione.* Livorno, R. Giusti, 1920. 8vo. 12 + 297 pp. L. 8.50
- CONCILIUS (E. DE). *Lezioni di meccanica applicata alle costruzioni. Parte 1: Introduzione. (R. Scuola superiore politecnica di Napoli.)* Napoli, tip. F. Lubrano, 1919. 8vo. 7 + 112 pp. L. 7.50
- GRASSI (G.). *Principi scientifici della elettrotecnica; introduzione al corso di elettrotecnica. 4a edizione. (Grande biblioteca tecnica, No. 7.)* Torino, soc. tip. ed. Nazionale, 1920. 8vo. 8 + 336 pp. L. 20.00
- JUNG (H.). See Nölke (F.).
- LEGRAND (L.). *Cours de mécanique rationnelle avec de nombreuses applications à l'usage des ingénieurs.* Paris et Liège, Béranger, 1920. 8vo. 364 pp. Fr. 48.00
- NÖLKE (F.). *Das Problem der Entwicklung unseres Planetensystems. Eine kritische Studie. 2te völlig umgearbeitete Auflage. Mit einem Geleitwort von H. Jung.* Berlin, Springer, 1919. M. 30.80
- OSSEN (C. W.). *Atomiska föreställningar i nutidens fysik. Tid, rum och materia. Femton föreläsningar.* Stockholm, Albert Bonniers Förlag, 1919.
- PEEL (T.). *Examples in heat and heat engines.* Cambridge, University Press, 1919. 3+104 pp. 5s.
- POLSTER (H.). *Kinematik.* Neudruck. Berlin, 1919. M. 1.25
- ROSE (W. N.). *Mathematics for engineers. Part 2.* London, Chapman and Hall, 1920. 8vo. 419 pp. 13s. 6d.
- SECRIST (H.). *Statistics in business.* New York, McGraw-Hill, 1919. 8vo. 9 + 137 pp.
- STEVENS (A. B.). *Arithmetic of pharmacy. 4th edition revised and enlarged.* New York, Van Nostrand, 1920. 8vo. 110 pp. \$1.50
- VINAL (E. R.). *Mathematics for the accountant.* New York, Biddle Publishing Company, 1920. 8vo. 16 + 172 pp. \$2.50

TWENTY-NINTH ANNUAL LIST OF PAPERS.

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND
SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO
THE PLACES OF THEIR PUBLICATION.

ALEXANDER, J. W. Note on two three-dimensional manifolds with the same group. Read Sept. 2, 1919. *Transactions of the American Mathematical Society*, vol. 20, No. 4, pp. 339-342; Oct., 1919.

— A proof of Jordan's theorem about a simple closed curve. Read April 29, 1916. *Annals of Mathematics*, ser. 2, vol. 21, No. 3, pp. 180-184; March, 1920.

— Note on Riemann spaces. Read Feb. 28, 1920. *Bulletin of the American Mathematical Society*, vol. 26, No. 8, pp. 370-372; May, 1920.

ALGER, P. L. See VEBLEN, O.

ALTSCHILLER-COURT (N.). On a pencil of nodal cubics. Read Dec. 31, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 5, pp. 203-211; Feb., 1920.

— On the orthocentric quadrilateral. Read Dec. 31, 1919. *American Mathematical Monthly*, vol. 27, No. 5, pp. 199-202; May, 1920.

BARNETT, I. A. Integro-differential equations with constant limits of integration. Read Dec. 28, 1918. *Bulletin of the American Mathematical Society*, vol. 26, No. 5, pp. 193-203; Feb., 1920.

BAUER, L. A. Geophysics at the Brussels meetings, July 18-28, 1919. Read Sept. 4, 1919. *Science*, new ser., vol. 50, No. 1296, pp. 399-403; Oct. 31, 1919.

BELL, E. T. The twelve elliptic functions related to sixteen doubly periodic functions of the second kind. Read (San Francisco) April 10, 1920. *Messenger of Mathematics*, new ser., vol. 49, Nos. 5-6, pp. 78-84; Sept. and Oct., 1919.

— On the number of representations of $2n$ as a sum of $2r$ squares. Read (San Francisco) April 5, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 1, pp. 19-25; Oct., 1919.

— Sur les représentations propres par quelques formes quadratiques de Liouville. Read (San Francisco) Oct. 25, 1919. *Comptes Rendus de l'Académie des Sciences*, vol. 169, No. 17, pp. 711-712; Oct. 27, 1919.

— On the enumeration of proper and improper representations in homogeneous forms. Read (San Francisco) Oct. 25, 1919. *Annals of Mathematics*, ser. 2, vol. 21, No. 3, pp. 166-179; March, 1920.

BENNETT, A. A. The sign of the distance in analytical geometry. Read Oct. 25, 1919. *American Mathematical Monthly*, vol. 26, No. 8, pp. 344-350; Oct., 1919.

— Poncelet polygons in higher space. Read Dec. 30, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 6, pp. 274-275; March, 1920.

- BIRKHOFF, G. D. Recent advances in dynamics. Read Dec. 30, 1919. *Science*, new ser., vol. 51, No. 1307, pp. 51-55; Jan. 16, 1920.
- BLISS, G. A. The use of adjoint systems in the problem of differential corrections for trajectories. Read March 29, 1919. *Journal of the United States Artillery*, vol. 51, No. 3, pp. 296-311; Sept., 1919.
- A method of computing differential corrections for a trajectory. Read March 29, 1919. *Journal of the United States Artillery*, vol. 51, No. 4, pp. 445-449; Oct., 1919.
- Differential equations containing arbitrary functions. Read Sept. 3, 1919. *Transactions of the American Mathematical Society*, vol. 21, No. 2, pp. 79-92; April, 1920.
- Functions of lines in ballistics. Read Sept. 3, 1919. *Transactions of the American Mathematical Society*, vol. 21, No. 2, pp. 93-106; April, 1920.
- Some recent developments in the calculus of variations. Read Dec. 30, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 8, pp. 343-361; May, 1920.
- BLUMBERG, H. On certain saltus equations. Read (Southwestern Section) Nov. 29, 1913, and Dec. 27, 1913. *American Journal of Mathematics*, vol. 41, No. 3, pp. 183-190; July, 1919.
- BORDEN, R. F. On the adjoint of a certain mixed equation. Read Dec. 30, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 9, pp. 408-412; June, 1920.
- BRINK, R. W. A new sequence of integral tests for the convergence and divergence of infinite series. Read April 29, 1916. *Annals of Mathematics*, ser. 2, vol. 21, No. 1, pp. 39-60; Sept., 1919.
- BUCHANAN, D. Asymptotic satellites near the equilateral-triangle equilibrium points in the problem of three bodies. Read Dec. 27, 1916. *Transactions of the Cambridge Philosophical Society*, vol. 22, No. 15, pp. 309-340; Oct., 1919.
- Periodic orbits on a surface of revolution. Read Dec. 26, 1913 and Sept. 5, 1918. *American Journal of Mathematics*, vol. 42, No. 1, pp. 47-75; Jan., 1920.
- CAIRNS, W. D. A derivation of the equation of the normal probability curve. Read Sept. 5, 1918. *Bulletin of the American Mathematical Society*, vol. 26, No. 3, pp. 105-108; Dec., 1919.
- Certain properties of binomial coefficients. Read Sept. 4, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 4, pp. 160-164; Jan., 1920.
- CARMICHAEL, R. D. On a general class of integrals of the form $\int_0^\infty \varphi(t)g(x+t)dt$. Read April 13, 1918. *Transactions of the American Mathematical Society*, vol. 20, No. 4, pp. 313-322; Oct., 1919.
- Note on convergence tests for series and on Stieltjes integration by parts. Read Sept. 4, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 3, pp. 97-102; Dec., 1919.
- Note on a physical interpretation of Stieltjes integrals. Read Sept. 4, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 3, pp. 102-105; Dec., 1919.

- Conditions necessary and sufficient for the existence of a Stieltjes integral. Read Sept. 4, 1919. *Proceedings of the National Academy of Sciences*, vol. 5, No. 12, pp. 551-555; Dec., 1919.
- On sequences of integers defined by recurrence relations. Read Dec. 29, 1917. *Quarterly Journal of Mathematics*, vol. 48, No. 4, pp. 343-372; Jan., 1920.
- On the convergence of certain classes of series of functions. Read Oct. 25, 1919. *American Journal of Mathematics*, vol. 42, No. 2, pp. 77-90; April 1920.
- CARVER, W. B., and KING, E. F. A property of permutation groups analogous to multiple transitivity. Read Dec. 31, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 7, pp. 319-322; April, 1920.
- CHAO, Y. R. A note on "continuous mathematical induction." Read (San Francisco) April 5, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 1, pp. 17-18; Oct., 1919.
- CHITTENDEN, E. W., and PITCHER, A. D. On the theory of developments of an abstract class in relation to the calcul fonctionnel. Read March 22, 1913, April 6, 1917 and March 28, 1919. *Transactions of the American Mathematical Society*, vol. 20, No. 3, pp. 213-233; July, 1919.
- COBLE, A. B. The ten nodes of the rational sextic and of the Cayley symmetroid. Read March 28, 1919. *American Journal of Mathematics*, vol. 41, No. 4, pp. 243-265; Oct., 1919.
- COOLIDGE, J. L. The geometry of hermitian forms. Read Dec. 30, 1919. *Transactions of the American Mathematical Society*, vol. 21, No. 1, pp. 44-51; Jan., 1920.
- CURTIS, M. F. On the rectifiability of a twisted cubic. Read Dec. 31, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 6, pp. 275-277; March, 1920.
- DANIELL, P. J. Integrals in an infinite number of dimensions. Read Dec. 28, 1918. *Annals of Mathematics*, ser. 2, vol. 20, No. 4, pp. 281-288; June, 1919.
- DECKER, F. F. On the order of a restricted system of equations. Read April 27, 1918. *American Journal of Mathematics*, vol. 41, No. 4, pp. 283-298; Oct., 1919.
- EIESLAND, J. Flat-sphere geometry, third paper. Read Sept. 4, 1916. *Tôhoku Mathematical Journal*, vol. 16, Nos. 3-4, pp. 185-235; Nov., 1919.
- EISENHART, L. P. Triply conjugate systems with equal point invariants. Read April 26, 1919. *Annals of Mathematics*, ser. 2, vol. 20, No. 4, pp. 262-273; June, 1919.
- Transformations of surfaces applicable to a quadric. Read Sept. 3, 1919. *Transactions of the American Mathematical Society*, vol. 20, No. 4, pp. 323-338; Oct., 1919.
- Transformations of cyclic systems of circles. Read Sept. 3, 1919. *Proceedings of the National Academy of Sciences*, vol. 5, No. 12, pp. 555-557; Dec., 1919.
- Darboux's Anteil an der Geometrie. Read Sept. 6, 1917. *Acta Mathematica*, vol. 42, No. 3, pp. 275-284; 1919.
- EMCH, A. On a certain class of rational ruled surfaces. Read Dec. 28, 1918. *Proceedings of the National Academy of Sciences*, vol. 5, No. 6, pp. 222-224; June, 1919.

- FORD, W. B. A brief account of the life and work of the late Professor Ulisse Dini. Read Sept. 3, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 4, pp. 173-177; Jan., 1920.
- FRANKLIN, P. Computation of the complex zeros of the function $P(z)$ complementary to the incomplete gamma function. Read April 26, 1919. *Annals of Mathematics*, ser. 2, vol. 21, No. 1, pp. 61-63; Sept., 1919.
- FRY, T. C. The solution of circuit problems. Mathematical methods of investigation resulting from the application of Fourier's integral. Read April 26, 1919. *Physical Review*, ser. 2, vol. 14, No. 2, pp. 115-136; Aug., 1919.
- GABA, M. G. A set of completely independent postulates for the linear order η . Read Sept. 4, 1917. *Bulletin of the American Mathematical Society*, vol. 26, No. 4, pp. 158-159; Jan., 1920.
- GLENN, O. E. On a new treatment of theorems of finiteness. Read Dec. 27, 1917 and Sept. 5, 1918. *Transactions of the American Mathematical Society*, vol. 20, No. 3, pp. 203-212; July, 1919.
- GREEN, G. M. Nets of space curves. Read April 24, 1915. *Transactions of the American Mathematical Society*, vol. 21, No. 2, pp. 207-236; April, 1920.
- GRONWALL, T. H. On the influence of keyways on the stress distribution in cylindrical shafts. Read Feb. 26, 1918. *Transactions of the American Mathematical Society*, vol. 20, No. 3, pp. 234-244; July, 1919.
- HALLETT, G. H. Linear order in three dimensional euclidean and double elliptic spaces. Read April 27, 1918. *Annals of Mathematics*, ser. 2, vol. 21, No. 3, pp. 185-202; March, 1920.
- HANCOCK, H. Les fondements de la théorie des fonctions elliptiques. Read Dec. 28, 1918. *Bulletin de la Société Mathématique de France*, vol. 47, Nos. 1-2, pp. 37-42; 1919.
- Remarks regarding the relations among the roots of the cubic and biquadratic equations. Read Dec. 28, 1918. *American Mathematical Monthly*, vol. 26, No. 7, pp. 291-295; Sept., 1919.
- HAZLETT, O. C. A theorem on modular covariants. Read Dec. 28, 1918. *Transactions of the American Mathematical Society*, vol. 21, No. 2, pp. 247-254; April, 1920.
- HOPKINS, J. W. Some convergent developments associated with irregular boundary conditions. Read April 27, 1918. *Transactions of the American Mathematical Society*, vol. 20, No. 3, pp. 245-259; July, 1919.
- HOSKINS, L. M. The strain of a gravitating sphere of variable density and elasticity. Read (San Francisco) Oct. 27, 1917 and (San Francisco) April 5, 1919. *Transactions of the American Mathematical Society*, vol. 21, No. 1, pp. 1-43; Jan., 1920.
- HUNTINGTON, E. V. Mathematics and statistics, with an elementary account of the correlation coefficient and the correlation ratio. Read Sept. 4, 1919. *American Mathematical Monthly*, vol. 26, No. 10, pp. 421-435; Dec., 1919.
- JAMES, G. On the theory of summability. Read Dec. 22, 1916. *Annals of Mathematics*, ser. 2, vol. 21, No. 2, pp. 120-127; Dec., 1919.
- JOHNSON, W. W. On Napier's circular parts. Read Sept. 5, 1918. *Messenger of Mathematics*, vol. 48, No. 10, pp. 145-153; Feb., 1919.

- KING, E. F. See CARVER, W. B.
- KLINE, J. R. Concerning sense on closed curves in non-metrical plane analysis situs. Read April 26, 1919. *Annals of Mathematics*, ser. 2, vol. 21, No. 2, pp. 113-119; Dec., 1919.
- LOVE, C. E. Note on a class of singular integral equations of the second kind. Read Dec. 30, 1919. *Annals of Mathematics*, ser. 2, vol. 21, No. 2, pp. 104-112; Dec., 1919.
- LUNN, A. C. Some functional equations in the theory of relativity. Read April 26, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 1, pp. 26-34; Oct., 1919.
- A principle of duality in thermodynamics. Read Dec. 31, 1919. *Physical Review*, ser. 2, vol. 15, No. 4, pp. 269-276; April, 1920.
- MCATEE, J. E. Modular invariants of a quadratic form for a prime power modulus. Read (Southwestern Section) Dec. 1, 1917. *American Journal of Mathematics*, vol. 41, No. 3, pp. 225-242; July, 1919.
- MC EWEN, G. F., and MICHAEL, E. L. The functional relation of one variable to each of a number of correlated variables determined by a method of successive approximation to group averages: a contribution to statistical methods. Read (San Francisco) April 7, 1917. *Proceedings of the American Academy of Arts and Sciences*, vol. 55, No. 2, pp. 95-133; Dec., 1919.
- MACNEISH, H. F. The sum of the face angles of certain polyhedrons in n -space. Read Dec. 30, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 9, pp. 398-399; June, 1920.
- MATHEWSON, L. C. On the groups of isomorphisms of a system of abelian groups of order p^m and type $(n, 1, 1, \dots, 1)$. Read Sept. 5, 1918. *American Journal of Mathematics*, vol. 42, No. 2, pp. 119-128; April, 1920.
- MICHAEL, E. L. See MC EWEN, G. F.
- MILLER, G. A. Groups possessing a small number of sets of conjugate operators. Read Dec. 28, 1918. *Transactions of the American Mathematical Society*, vol. 20, No. 3, pp. 260-270; July, 1919.
- Form of the number of subgroups of prime power groups. Read Sept. 3, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 2, pp. 66-72; Nov., 1919.
- Groups generated by two operators of order three whose product is of order four. Read Dec. 30, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 8, pp. 361-369; May, 1920.
- MOORE, C. L. E. Surfaces of rotation in a space of four dimensions. Read April 27, 1918. *Annals of Mathematics*, ser. 2, vol. 21, No. 2, pp. 81-93; Dec., 1919.
- Rotation surfaces of constant curvature in space of four dimensions. Read April 24, 1920. *Bulletin of the American Mathematical Society*, vol. 26, No. 10, pp. 454-460; July, 1920.
- MOORE, C. L. E., and PHILLIPS, H. B. Rotations in space of even dimensions. Read Dec. 30, 1919. *Proceedings of the American Academy of Arts and Sciences*, vol. 55, No. 4, pp. 155-188; March, 1920.
- Note on geometrical products. Read Dec. 30, 1919. *Proceedings of the National Academy of Sciences*, vol. 6, No. 3, pp. 155-158; March, 1920.

- MOORE, C. N. On the summability of the developments in Bessel's functions. Read Sept. 4, 1917 and April 12, 1918. *Transactions of the American Mathematical Society*, vol. 21, No. 2, pp. 107-156; April, 1920.
- MOORE, R. L. On the most general class L of Fréchet in which the Heine-Borel-Lebesgue theorem holds true. Read April 26, 1919. *Proceedings of the National Academy of Sciences*, vol. 5, No. 6, pp. 206-210; June, 1919.
- On the Lie-Riemann-Helmholtz-Hilbert problem of the foundations of geometry. Read Dec. 28, 1915. *American Journal of Mathematics*, vol. 41, No. 4, pp. 299-319; Oct., 1919.
- MORLEY, F. On the Lüroth quartic curve. Read Dec. 28, 1916. *American Journal of Mathematics*, vol. 41, No. 4, pp. 279-282; Oct., 1919.
- MOULTON, F. R. On the stability of direct and retrograde satellite orbits. Read April 11, 1914. *Monthly Notices of the Royal Astronomical Society*, vol. 75, No. 2, pp. 40-57; Dec., 1914.
- PELL, A. J. A general system of linear equations. Read Dec. 28, 1917. *Transactions of the American Mathematical Society*, vol. 20, No. 4, pp. 343-355; Oct., 1919.
- PHILLIPS, H. B. Functions of matrices. Read April 27, 1918. *American Journal of Mathematics*, vol. 41, No. 4, pp. 266-278; Oct., 1919.
- See MOORE, C. L. E.
- PITCHER, A. D. See CHITTENDEN, E. W.
- REYNOLDS, C. N. Note on linear differential equations of the fourth order whose solutions satisfy a homogeneous quadratic identity. Read Dec. 31, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 6, pp. 277-280; March, 1920.
- RIETZ, H. L. On functional relations for which the coefficient of correlation is zero. Read March 29, 1919. *Quarterly Publications of the American Statistical Association*, new ser., vol. 16, No. 127, pp. 472-476; Sept., 1919.
- ROBINSON, L. B. Un système symétrique de polynômes. Read Sept. 4, 1918 and April 26, 1919. *Comptes Rendus de l'Académie des Sciences*, vol. 169, No. 12, pp. 524-526; Sept. 22, 1919.
- SAFFORD, F. H. Reduction of the elliptic element to the Weierstrass form. Read April 26, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 1, pp. 13-16; Oct., 1919.
- Parametric equations of the path of a projectile when the air resistance varies as the n th power of the velocity. Read April 27, 1918. *Bulletin of the American Mathematical Society*, vol. 26, No. 7, pp. 289-293; April, 1920.
- SHARPE, F. R., and SNYDER, V. Certain types of involutorial space transformations. Read April 27, 1918. *Transactions of the American Mathematical Society*, vol. 20, No. 3, pp. 185-202; July, 1919.
- Certain types of involutorial space transformations. Read Sept. 3, 1919. *Transactions of the American Mathematical Society*, vol. 21, No. 1, pp. 52-78; Jan., 1920.
- SHUGERT, S. P. The resolvents of König and other types of symmetric functions. Read Sept. 3, 1919. Author's dissertation. Lancaster, 1919. 19 pp.

- SILVERMAN, L. L. On the consistency and equivalence of certain generalized definitions of the limit of a function of a continuous variable. Read Dec. 28, 1915. *Annals of Mathematics*, ser. 2, vol. 21, No. 2, pp. 128-140; Dec., 1919.
- SISAM, C. H. On surfaces containing two pencils of cubic curves. Read Dec. 28, 1918. *American Journal of Mathematics*, vol. 41, No. 3, pp. 212-224; July, 1919.
- SMAIL, L. L. A general method for the summation of divergent series. Read April 26, 1913 and (San Francisco) May 22, 1914. *Annals of Mathematics*, ser. 2, vol. 20, No. 2, pp. 149-154; Dec., 1918.
- Summability of double series. Read (San Francisco) April 5, 1919. *Annals of Mathematics*, ser. 2, vol. 21, No. 3, pp. 221-223; March, 1920.
- SNYDER, V. See SHARPE, F. R.
- TAYLOR, J. S. Sheffer's set of five postulates for boolean algebras in terms of the operation "rejection" made completely independent. Read Dec. 31, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 10, pp. 449-454; July, 1920.
- VANDIVER, H. S. A property of cyclotomic integers and its relation to Fermat's last theorem. Read Jan. 2, 1915. *Annals of Mathematics*, ser. 2, vol. 21, No. 2, pp. 73-80; Dec., 1919.
- VAN VLECK, E. B. On the combination of non-loxodromic substitutions. Read April 22, 1916. *Transactions of the American Mathematical Society*, vol. 20, No. 4, pp. 299-312; Oct., 1919.
- VEBLEN, O., and ALGER, P. L. Rotating bands. Read April 26, 1919. *Journal of the United States Artillery*, vol. 51, No. 4, pp. 355-390; Oct., 1919.
- WALSH, J. L. On the proof of Cauchy's integral formula by means of Green's formula. Read Dec. 30, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 4, pp. 155-157; Jan., 1920.
- On the solution of linear equations in infinitely many variables by successive approximations. Read April 7, 1917. *American Journal of Mathematics*, vol. 42, No. 2, pp. 91-96; April, 1920.
- WEAR, L. E. Self-dual plane curves of the fourth order. Read (San Francisco) May 22, 1914. *American Journal of Mathematics*, vol. 42, No. 2, pp. 97-118; April, 1920.
- WEAVER, J. H. Some extensions of the work of Pappus and Steiner on tangent circles. Read April 29, 1916. *American Mathematical Monthly*, vol. 27, No. 1, pp. 2-11; Jan., 1920.
- WEBSTER, A. G. Acoustical impedance, and the theory of horns and of the phonograph. Read Sept. 4, 1916. *Proceedings of the National Academy of Sciences*, vol. 5, No. 7, pp. 275-282; July, 1919.
- WIENER, N. Bilinear operations generating all operations in a domain Ω . Read Dec. 30, 1919. *Annals of Mathematics*, ser. 2, vol. 21, No. 3, pp. 157-165; March, 1920.
- A set of postulates for fields. Read Dec. 30, 1919. *Transactions of the American Mathematical Society*, vol. 21, No. 2, pp. 237-246; April, 1920.
- WILCZYŃSKI, E. J. In memory of Gabriel Marcus Green. Read March 28, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 1, pp. 1-13; Oct., 1919.

- Line-geometric representations for functions of a complex variable. Read Dec. 28, 1918 and March 28, 1919. *Transactions of the American Mathematical Society*, vol. 20, No. 4, pp. 271-298; Oct., 1919.
- One-parameter families and nets of ruled surfaces and a new theory of congruences. Read Dec. 31, 1919. *Transactions of the American Mathematical Society*, vol. 21, No. 2, pp. 157-206; April, 1920.
- WILSON, W. H. On certain related functional equations. Read Dec. 27, 1917. *Bulletin of the American Mathematical Society*, vol. 26, No. 7, pp. 300-312; April, 1920.
- WINGER, R. M. Some generalisations of the satellite theory. Read (San Francisco) Oct. 25, 1919. *Bulletin of the American Mathematical Society*, vol. 26, No. 2, pp. 75-79; Nov., 1919.
- On the satellite line of the cubic. Read (San Francisco) April 6, 1918. *American Journal of Mathematics*, vol. 42, No. 2, pp. 129-135; April, 1920.

INDEX OF VOLUME XXVI.

- ALEXANDER, J. W. Note on Riemann Spaces, 370.
- ALLEN, F. E. See REVIEWS, under Wentworth.
- ALTSHILLER-COURT, N. On a Pencil of Nodal Cubics, 203.
- BARNETT, I. A. Integro-Differential Equations with Constant Limits of Integration, 193.
- BELL, E. T. On the Number of Representations of $2n$ as a Sum of $2r$ Squares, 19.
- Definition and Illustrations of New Arithmetical Group Invariants, 211.
- The Equation $ds^2 = dx^2 + dy^2 + dz^2$, 312.
- BENNETT, A. A. Poncelet Polygons in Higher Space, 274.
- An Acknowledgment of Priority, 280.
- BERNSTEIN, B. A. Reports of Meetings of the San Francisco Section: October Meeting, 152; April Meeting, 396.
- BIRKHOFF, G. D. See REVIEWS, under Poincaré, Whittaker.
- BLISS, G. A. Some Recent Developments in the Calculus of Variations, 343.
- BORDEN, R. F. On the Adjoint of a certain Mixed Equation, 408.
- CAIRNS, W. D. A Derivation of the Equation of the Normal Probability Curve, 105.
- Certain Properties of Binomial Coefficients, 160.
- CARMICHAEL, R. D. Note on Convergence Tests for Series and on Stieltjes Integration by Parts, 97.
- Note on a Physical Interpretation of Stieltjes Integrals, 102, 136.
- See REVIEWS, under Ahrens, Castelnuovo, de Mendizabal Tamborrel.
- CARVER, W. B. and KING, E. F. A Property of Permutation Groups Analogous to Multiple Transitivity, 319.
- CHAO, Y. R. A Note on "Continuous Mathematical Induction," 17.
- COLE, F. N. Reports of Meetings of the American Mathematical Society: October Meeting, 145; Twenty-Sixth Annual Meeting, 241; February Meeting, 337; April Meeting, 433.
- COWLEY, E. B. See REVIEWS, under Brenke, Merrill, Skinner.
- CRAIG, C. F. See REVIEWS, under McClenon.
- CRATHORNE, A. R. See REVIEWS, under Hancock, Running.
- CURRIER, C. H. See REVIEWS, under Clements, Skinner.
- CURTIS, M. F. On the Rectifiability of a Twisted Cubic, 275.
- DANIELL, P. J. Stieltjes Derivatives, 444.
- DINI, U. See FORD, W. B.
- DOWLING, L. W. See REVIEWS, under Browne.

- DRESDEN, A. Report of the April Meeting of the American Mathematical Society at Chicago, 385.
- DRESDEN, A., and KELLOGG, O. D. Report of the St. Louis Meeting of the American Mathematical Society, 260.
- FORD, W. B. A Brief Account of the Life and Work of the Late Professor Ulisse Dini, 173.
- FORSYTH, C. H. Formulas for Constructing Abridged Mortality Tables for Decennial Ages, 34, 135.
- See REVIEWS, under Sutton.
- GABA, M. G.. A Set of Completely Independent Postulates for the Linear Order η , 158.
- GREEN, G. M. See WILCZYNSKI, E. J.
- HAYASHI, T. On the Rectifiability of a Twisted Cubic, 73, 136.
- HILDEBRANDT, T. H. On Bordered Fredholm Determinants, 400.
- KARPINSKI, L. C. See REVIEWS, under Archibald.
- KELLOGG, O. D. See DRESDEN, A.
- KING, E. F. See CARVER, W. B.
- LEHMER, D. N. See REVIEWS, under Dickson.
- LUNN, A. C. Some Functional Equations in the Theory of Relativity, 26.
- MACNEISH, H. F. The Sum of the Face Angles of certain Polyhedrons in n -Space, 398.
- MILLER, G. A. Form of the Number of Subgroups of Prime Power Groups, 66.
- Groups Generated by Two Operators of Order Three whose Product is of Order Four, 361.
- See REVIEWS, under Cajori.
- MILNE, W. E. Infinite Systems of Functions, 294.
- MOORE, C. L. E. Rotation Surfaces of Constant Curvature in Space of Four Dimensions, 454.
- MOORE, R. L. See REVIEWS, under Veblen.
- MOULTON, E. J. Report of the Twenty-Sixth Summer Meeting of the American Mathematical Society, 49.
- OWENS, F. W. See REVIEWS, under Dowling.
- PITCHER, A. D. On the Coherence of Certain Systems in General Analysis, 405.
- REYNOLDS, C. N., JR. Note on Linear Differential Equations of the Fourth Order whose Solutions Satisfy a Homogeneous Quadratic Identity, 277.
- RICHARDSON, R. G. D. See REVIEWS, under Böcher.
- SAFFORD, F. H. Reduction of the Elliptic Element to the Weierstrass Form, 13.
- Parametric Equations of the Path of a Projectile when the Air Resistance Varies as the n th Power of the Velocity, 289.
- See REVIEWS, under Lepper.
- SHAW, J. B. See REVIEWS, under Cullis.

- SLOCUM, S. E. See *REVIEWS*, under Taylor, Wentworth.
- TAYLOR, J. S. Sheffer's Set of Five Postulates for Boolean Algebras in Terms of the Operation "Rejection" Made Completely Independent, 449.
- WALSH, J. L. On the Proof of Cauchy's Integral Formula by Means of Green's Formula, 155.
- WILCZYŃSKI, E. J. In Memory of Gabriel Marcus Green, 1.
- WILSON, W. H. On Certain Related Functional Equations, 300.
- WINGER, R. M. Some Generalizations of the Satellite Theory, 75.

REVIEW.

- Ahrens, W. *Mathematische Spiele* (dritte Auflage), R. D. CARMICHAEL, 86.
- Allen, F. See Taylor, E. H.
- Archibald, R. C. *The Training of Teachers of Mathematics for the Secondary Schools. With the Editorial Cooperation of D. E. Smith, W. F. Osgood, and J. W. A. Young, L. C. KARPINSKI*, 179.
- Bôcher, M. *Leçons sur les Méthodes de Sturm dans la Théorie des Equations Différentielles linéaires et leur Développement modernes, recueillies et rédigées par G. Julia, R. G. D. RICHARDSON*, 108.
- Brenke, W. C. *Advanced Algebra*, E. B. COWLEY, 323.
- Brown, J. C. See Wentworth, G.
- Browne, R. T. *The Mystery of Space*, L. W. DOWLING, 460.
- Cajori, F. *A History of Mathematics*, G. A. MILLER, 79.
- Castelnuovo, G. *Calcolo delle Probabilità*, R. D. CARMICHAEL, 132.
- Clements, G. R. *Problems in the Mathematical Theory of Investment*, C. H. CURRIER, 463.
- Cullis, C. E. *Matrices and Determinoids*, volumes 1 and 2, J. B. SHAW, 224.
- Dickson, L. E. *History of the Theory of Numbers*, volume 1, D. N. LEHMER, 125, 281.
- Dowling, L. W. *Projective Geometry*, F. W. OWENS, 39.
- Hancock, H. *Theory of Maxima and Minima*, A. R. CRATHORNE, 180.
- Julia, G. See Bôcher, M.
- Lennes, N. J. See Sutton, C. W.
- Lepper, G. H. *From Nebula to Nebula* (fourth edition), F. H. SAFFORD, 182.
- McClenon, R. B. and Rusk, W. J. *Introduction to the Elementary Functions*, C. F. CRAIG, 40.
- Mendizabal Tamborrel, J. de. *Tratado Elemental de Goniometria* (segunda edición), R. D. CARMICHAEL, 86.
- Merrill, H. A. and Smith, C. E. *A First Course in Higher Algebra*, E. B. COWLEY, 323.
- Osgood, W. F. See Archibald, R. C.

- Poincaré, H. *Oeuvres*, tome 2, G. D. BIRKHOFF, 164.
Running, T. R. *Empirical Formulas*, A. R. CRATHORNE, 376.
Rusk, W. J. See McClenon, R. B.
Schlauch, W. S. See Wentworth, G.
Skinner, E. B. *College Algebra*, E. B. COWLEY, 323.
—Tables from the *Mathematical Theory of Investment*, C. H. CURRIER, 463.
Smith, C. E. See Merrill, H. A.
Smith, D. E. See Archibald, R. C., Wentworth, G.
Sutton, C. W., and Lennes, N. J. *Business Arithmetic*, C. H. FORSYTH, 375.
Taylor, E. H., and Allen, F. *Junior High School Mathematics*, books 1 and 2, S. E. SLOCUM, 462.
Veblen, O., and Young, J. W. *Projective Geometry*, volume 2, R. L. MOORE, 412.
Wentworth, G., Smith, D. E., and Brown, J. C. *Junior High School Mathematics*, S. E. SLOCUM, 373.
Wentworth, G., Smith, D. E., and Schlauch, W. S. *Commercial Algebra*, F. E. ALLEN, 177.
Whittaker, E. T. *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies; with an Introduction to the Problem of Three Bodies* (second edition), G. D. BIRKHOFF, 183.
Young, J. W. See Veblen, O.
Young, J. W. A. See Archibald, R. C.
-

Corrections, 135, 380.

Index of Volume XXVI, 480.

New Publications, 46, 94, 142, 189, 237, 285, 334, 380, 430, 469.

Notes, 41, 87, 136, 184, 233, 281, 329, 377, 425, 464.

Papers Read before the Society and Subsequently Published, Twenty-Ninth Annual List of, 472.

NOTES AND OTHER ITEMS.

Academies, Associations, Congresses, and Societies:

American Mathematical Society: Annual Meeting, 136; Committee on Bibliography, 242, 435; Election of Officers, 243; Incorporation, 242, 434; List of Officers and Members, 136; New Members Admitted, 49, 145, 242, 337, 434; St. Louis Meeting, 136; San Francisco Section, June Meeting, 396; Statistics, 243; Summer Meeting and Colloquium, 329; Transactions, 87, 184, 234, 426, 434.

Associations for the Advancement of Science, American, 136, 233, 377, 467; British, 377.

American Astronomical Society, 87; Astronomical Society of the Pacific, 283; Belgian Academy of Sciences, 138; Edinburgh Mathematical Society, 184, 331, 464; Göttingen Society of Sciences, 331;

International Mathematical Union, 184, 337, 425, 435, 464; London Mathematical Society, 87, 330; Mathematical Association of America, 87, 136, 233, 234, 425; Mathematical Association of Japan, 282; Mathematical Society of Greece, 377; National Academy of Sciences, 187, 426; National Research Council, 184, 337, 425; Norwegian Mathematical Society, 42; Paris Academy of Sciences, 138, 282; Prince Jablonowski Society, 332; Royal Institute of Venice, 91; Royal Society of London, 185, 282; Strasbourg Congress, 330, 464.

Catalogues of Books, 188, 237, 469.

Doctorates in Mathematics, American, 137.

Journals: American Journal of Mathematics, 42, 234, 330, 464; Annals of Mathematics, 41, 136, 281, 426; Bulletin of the Mathematical Society of Greece, 377; Bulletin of the Norwegian Mathematical Society, 42; L'Education Mathématique, 138; Giornale di Matematica Finanziaria, 282; Jahrbuch über die Fortschritte der Mathematik, 235; Journal of the Mathematical Association of Japan, 282; Mathematische Annalen, 465; Mathematische Zeitschrift, 42; Proceedings of the National Academy of Sciences, 41; Revue de Mathématiques Spéciales, 138; Transactions of the American Mathematical Society, 87, 184, 234, 426, 434.

Papers and Communications Presented to the Society, Authors:

Alexander, J. W., 52, 338, 338.
 Alger, P. L., 244.
 Allen, E. S., 52.
 Althiller-Court, N., 262, 262, 435.
 Barnett, I. A., 435, 435.
 Bell, E. T., 152, 152, 152, 396, 396, 396.
 Bennett, A. A., 146, 146, 146, 146, 244, 244, 338.
 Birkhoff, G. D., 146, 261, 436.
 Blichfeldt, H. F., 396.
 Bliss, G. A., 53, 53, 261.
 Borden, R. F., 261.
 Borger, R. L., 262.
 Bouton, C. L., 245.
 Boutroux, P., 262, 338.
 Brink, R. W., 52.
 Brown, E. W., 50.
 Buchanan, D., 245.
 Cairns, W. D., 53.
 Cajori, F., 152, 396.
 Camp, B. H., 338.
 Carmichael, R. D., 53, 53, 53, 146, 146, 143, 338, 338, 338.
 Carver, W. B., 245.
 Coble, A. B., 262, 262, 338.
 Cohen, T., 245.
 Coolidge, J. L., 244.
 Curtis, M. F., 245.
 Curtiss, D. R., 53, 338.
 Dantzig, T., 435.
 Dickson, L. E., 435.
 Dines, L. I., 53.
 Dodd, E. L., 262.

Douglas, J., 435.
 Eisenhart, L. P., 53, 53, 435.
 Emch, A., 262.
 Ettlinger, H. J., 52, 262, 262.
 Evans, G. C., 53, 262.
 Field, P., 52.
 Fischer, C. A., 245.
 Fite, W. B., 245.
 Ford, L. R., 435.
 Ford, W. B., 52.
 Forsyth, C. H., 245, 245.
 Fort, T., 262.
 Glenn, O. E., 244, 244, 338.
 Graustein, W. C., 245, 245.
 Gronwall, T. H., 146, 338, 338.
 Hardy, G. H., 338.
 Haskell, M. W., 152.
 Harlett, O. C., 146, 245.
 Hildebrandt, T. H., 53.
 Hollcroft, T. R., 435.
 Hoskins, L. M., 396.
 Howe, H. A., 53.
 Howland, L. A., 244.
 Huntington, E. V., 50.
 Jackson, D., 244, 338.
 James, G. O., 261.
 Kasner, E., 146, 338.
 Kellogg, O. D., 146, 436.
 Kempner, A. J., 52, 53.
 Keyser, C. J., 244.
 King, E. F., 245.
 Kline, J. R., 339.
 Lane, E. P., 52.
 Lefschets, S., 262.
 Linfield, B. Z., 261.

- Lipka, J., 244, 338, 436, 436.
 Love, C. E., 261.
 Lunn, A. C., 262, 386.
 McEwen, G. F., 262, 263.
 MacMillan, W. D., 388.
 MacNeish, H. F., 243.
 Manning, W. A., 152, 152.
 Mason, M., 386.
 Mason, T. E., 388.
 Miller, G. A., 52, 262, 262, 388.
 Moore, C. L. E., 244, 244, 433.
 Moore, E. H., 389.
 Morse, H. C. M., 244.
 Mullikin, A. M., 146.
 Nelson, A. L., 52.
 Page, L., 435.
 Pell, A. J., 146.
 Pfeiffer, G. A., 244.
 Phillips, H. B., 244, 244.
 Platt, H. H., 261.
 Post, E. L., 435, 435.
 Reynolds, C. N., Jr., 52, 52, 245.
 Rice, L. H., 52.
 Rider, P. R., 388.
 Rietz, H. L., 53, 262.
 Ritt, J. F., 435.
 Roe, E. D., Jr., 52.
 Roever, W. H., 53, 389.
 Rowe, J. E., 245, 436.
 Rutledge, G., 388.
 Sakellariou, N., 436.
 Schlesinger, F., 50.
 Schmiedel, O., 261.
 Schweitzer, A. R., 338, 338, 435.
 Sharpe, F. R., 52.
 Shaw, J. B., 53.
 Shugert, S. P., 52.
 Simonds, E. F., 146.
 Snyder, V., 52.
 Steimley, L. L., 262.
 Stouffer, E. B., 388.
 Taylor, J. S., 245.
 Vandiver, H. S., 436.
 Veblen, O., 436.
 Wahlin, G. E., 388.
 Walsh, J. L., 244, 244, 245.
 Wedderburn, J. H. M., 338.
 Whittemore, J. K., 146, 435.
 Wiener, N., 244, 244, 244, 244.
 Wilosynski, E. J., 262, 388.
 Williams, K. P., 262, 388.
 Winger, R. M., 152.
 Zeldin, S. D., 245.

Personal Notes:

Alasia de Quesada, C., 45; Albright, B., 284; Alexander, J. W., 467;
 Allen, E., 235; Allen, E. S., 140; Ames, L. D., 235; Anderegg, F., 467;
 Andoyer, H., 44; Anning, N., 236; Appell, P., 44, 429; Arms, R. A.,
 187; Atwood, O., 141.

Babbitt, A. E., 468; Baier, F. K., 467; Baire, R., 138; Baker, A.,
 333; Baldus, R., 185; Ballantyne, J. P., 333; Barnays, P., 185; Barnett,
 I. A., 430; Barney, I., 92; Barton, R. M., 235; Beal, F. W., 92; Bell,
 R. J. T., 283; Bennett, A. A., 140; Berwick, W. E. H., 466; Bersolari,
 L., 235; Bianchi, L., 379; Birkeland, 42; Birkhoff, G. D., 91, 136,
 233, 329, 379, 425, 434; Blaess, V., 185; Blankenstein, E. C., 140;
 Blaschke, W., 42, 185; Blichfeldt, H. F., 426; Blas, G. A., 136, 377;
 Blumberg, H., 45; Böcher, M., 140; Boersch, A., 469; Böhmer, P. E.,
 185; Boquet, F., 38; Borden, R. F., 379; Borgmeyer, C. J., 285; Born,
 M., 378; Bortolotti, E., 90; Böttcher, E., 188; Boussinesq, V. J., 139;
 Boutroux, P., 91; Boyd, P. P., 283; Bradshaw, J. W., 140; Bragg, W.
 H., 466, 466; Brahana, H. R., 430; Bramble, C. C., 186; Brand, L.,
 137; Brandenberger, C., 45; Brewster, J. A., 333; Brink, R. W., 467;
 Bromwich, T. J. I., 87; Brouwer, L. E., 465; Brown, E. W., 283;
 Brown, F. L., 187; Bruce, R. E., 283; Brunn, A. von, 378; Burch-
 nall, L. L., 283; Bussey, W. H., 467; Butterfield, A. D., 380.

Cairns, W. D., 467; Camp, C. C., 284; Campbell, A. D., 92;
 Campbell, J. E., 87; Campbell, J. W., 187; Cantor, M., 380, 465;
 Carathéodory, C., 465; Carmichael, R. D., 234; Carpenter, A. F.,
 185; Castelnovo, G., 90; Chapelon, 139; Chapman, F. E., 93; Chap-
 man, S., 90; Charbonnier, P., 138; Charles, R. L., 187; Chazy, J.,
 44; Clairin, J., 44; Collet, 139; Collier, M., 140; Colpitts, M. A., 93;
 Conwell, H. H., 187; Cowan, E. E., 430; Creese, G. H., 137; Crocco,

A., 235; Cummings, L. D., 141; Curtis, H. B., 187; Curtiss, D. R., 233, 377, 429; Czauber, E., 185.

Daniele, E., 235; Daniells, M. E., 187; Dantsig, T., 93; Danzer, O., 188; Davis, H. N., 140; Davis, J. E., 284; Dedekind, J. W. R., 235; De Donder, T., 184, 282; De Lury, A. T., 333; Demartres, G., 188; Demoulin, A., 138; Denjoy, A., 44; De Porte, J. V., 187; Dickson, L. E., 425, 430, 464; Dotterer, J. E., 284; Douglas, J., 468; Dresden, A., 186; Ducker, O. W., 284; Dumont, E., 45; Dunkel, O., 141; Duval, E. P. R., 430; Dziobek, O., 188.

Eddington, A. S., 138, 377; Ehrman, C. D., 284; Einstein, A., 332, 467; Eisenhart, L. P., 434, 464; Emch, A., 91; Epstein, P., 185; Everett, J. R., 468.

Fairon, J., 139; Falckenberg, H., 378; Fehr, H., 43; Feltges, E. M. F., 284; Feussner, W., 43; Field, P., 140; Fisher, G. E., 380; Flamme, D. J. B., 466; Flexner, S., 233; Floquet, A. M. G., 139; Forsyth, C. H., 186; Fraleigh, P. A., 284; Frege, G., 43; Furukawa, H., 282.

Garnier, R., 466; Garretson, W. V. N., 92; Gau, E., 139; Geer, P. van, 468; Gibbens, G., 467; Giraud, G., 138, 466; Glover, J. W., 91; Gob, A., 188; Godeaux, L., 138; Gossard, H. C., 186; Got, L. A. T., 139; Gouwnes, C., 468; Graber, M. E., 45; Graefe, F., 188; Gronwall, T. H., 91; Gross, W., 45; Grove, V. G., 284; Grüneisen, E., 43; Guichard, C., 44; Guimaraes, R., 188.

Haag, J., 139; Hadamard, J., 332, 379, 466; Hadley, L., 186; Haid, M., 468; Hardy, G. H., 87, 235; Harris, R. A., 140; Harrison, W. J., 332; Hart, W. L., 467; Hassé, H. R., 44; Hayashi, T., 282; Haslett, O. C., 467; Heath, T. L., 466; Hebbert, C. M., 187; Hecke, E., 185; Hedrick, E. R., 234; Heger, R., 469; Heinz, A. H., 283; Hemke, P., 187; Herdman, W. A., 377; Hess, G. W., 468; Hessenberg, G., 43, 139; Hilbert, D., 332; Ho, Y. H., 333; Hodge, F. H., 186; Holder, F. J., 141; Holgate, T. F., 140; Howe, A. M., 468; Humbert, P., 44; Huntington, E. V., 44, 91, 377; Hurwitz, A., 236; Husson, A. S. E., 466.

Ingraham, M., 284.

Jackson, D., 377; Jackson, T. W., 236; Jacobs, J. M., 137; Jacobsthal, E., 43; James, G., 186; James, G. D., 236; Janet, 139; Jeans, J. H., 185, 282; Johnson, M. F., 140; Jones, J. L., 333; Jordan, C., 426; Jourdain, P. E. B., 188; Jung, H., 466.

Kaba, M., 282; Kampé de Fériet, J., 139; Karpinski, L. C., 140; Kazarinoff, D. C., 140; Keller, J., 45; Kellogg, O. D., 233, 379; Kells, L. M., 93; Kent, F. C., 236; Kerr, F. L., 187; Killing, W., 139; Kircher, E. A. T., 92; Klein, F., 42, 43; Kline, J. R., 91; Knebelman, M. S., 93; Knopp, K., 185; Koenigs, G., 184; Kostka, C., 42; Krathwohl, W. C., 141; Krause, J. M., 185, 378; Kuroda, M., 282; Küstermann, W. W., 91.

Lamb, H., 466; Landis, W. W., 333; Lane, E. P., 92; Langman, H., 187; Larmor, J., 379; Lattès, S., 44; Laue, M. von, 378; Laura, E., 90; Leau, L., 139, 466; Lebesgue, H., 44, 138; Lefschetz, S., 282,

430, 466; LeSturgeon, F. E., 93; Levi, F., 185; Levi-Civita, T., 90; Lewis, F. P., 186; Lietzmann, W., 42; Lipka, J., 44; Loewy, A., 139; Longley, W. R., 283; Lorey, W., 43; Luck, J. J., 186; Lufkin, F. M., 284.

McCain, G. I., 137; McClenon, R. B., 186; McKelvey, J. V., 93, 430; McMahon, J., 333; MacLaurin, R. C., 237; MacLay, J., 188; MacLean, N. B., 233; MacMahon, P. A., 185; MacMillan, W. D., 140; MacNeish, H. F., 187; Maddox, A. C., 92; Malstroem, R., 469; Mangoldt, H. von, 43; Manning, H. P., 379; Mansion, P., 46; Mason, T. E., 186; Mathews, R. M., 141; Meacham, E. D., 186; Meek, H. B., 468; Merlin, E. L. A., 139; Merrill, H. A., 234; Miller, G. A., 44, 377; Miller, J. S., 284; Miller, K. A., 92; Millikan, R. A., 466; Millosewicz, E., 236; Mills, C. N., 186; Milne, W. E., 141; Miser, W. L., 141; Mises, R. von, 185, 465; Mohrmann, A., 43; Monroe, L. M., 92; Morse, D. S., 333; Morse, H. C. M., 284; Moulton, F. R., 44, 233, 329, 377; Muir, T., 429; Musselman, J. R., 141, 468.

Nagao, S., 282; Nelson, C. A., 430; Netto, E., 188; Noble, C. A., 141; Noether, E., 185; Nörlund, N. E., 43.

Olbrich, W., 378; Olson, H. L., 236.

Page, J. M., 377; Palmström, A., 42, 379; Parsons, F. W., 187; Pellet, A. C. E., 466; Perron, O., 332; Peterson, O. J., 140; Petrowitch, N., 184; Phalen, H. R., 92; Pierce, T. A., 92; Pincherle, S., 90; Pitcher, A. D., 377; Plemelj, J., 282; Pochhammer, L., 332; Poincaré, L., 429; Pollard, S., 379; Poritsky, H., 333; Pounder, I. R., 333; Pratt, A. S., 333; Proudman, J., 235; Putnam, T. M., 91.

Radon, J., 185; Raiford, T. S., 236; Ramanujan, S., 468; Ramsey, A., 284; Rankin, W. W., 45; Ranum, A., 333; Rayleigh, Lord, 45, 332; Rechart, F. E., 284; Reed, F. W., 92; Reina, V., 184, 236; Reye, K. T., 188; Reynolds, C. N., 137, 430; Rice, L. H., 236; Riets, H. L., 377; Robison, G. M., 92; Roever, W. H., 377; Roman, I. A., 93; Rowe, J. E., 379, 467; Roy, L., 44.

Sakellariou, N., 377; Schmeidler, M., 185; Schmidt, E., 43; Schur, F., 42, 139; Schur, I., 332, 465; Sensenig, W., 137; Servais, C., 282; Sharpe, F. R., 333; Sherk, W. H., 236; Sherwood, G. E. F., 140; Shively, L. S., 236; Shugert, S. P., 93; Shumway, R. R., 141; Simon, W. G., 430; Simpson, T. McN., 236; Sire, L., 466; Slaughter, H. E., 234; Smith, D. E., 234; Smith, H. L., 93; Smith, M. G., 92; Smith, P. F., 434; Snow, C., 380; Snyder, V., 44; Solberg, 42; Sousley, C. P., 284; Sperry, M. J., 468; Stäckel, P., 469; Stebbins, J., 377; Stephens, R. P., 141; Stickelberger, L., 139; Störmer, F. C. M., 42; Strutt, J. W., 45, 332; Sturm, R., 188; Stuyvaert, M. L. M., 139; Suppant-schitsch, R., 282.

Tanzola, J. J., 188; Tauber, A., 379; Taylor, E., 93; Taylor, J. M., 141; Thomson, J. J., 235, 283; Tietse, H., 139; Timerding, H. E., 43; Timpe, A., 43; Torrey, M. M., 468; Torroja, E., 93; Trayler, I. W., 284; Trefftz, E., 378; Tripp, M. O., 45; Trowbridge, A., 466; Tsao, C. C., 137; Turrière, E. L. F., 139.

Underhill, A. L., 467; Underwood, R. S., 236; Upton, C. B., 235.

Vallée Poussin, C. de la, 184; Vance, E. H., 91; Vandiver, H. S., 92; Van Vleck, E. B., 333; Veblen, O., 425; 434; Vessiot, E., 44; Voigt, W., 380; Volterra, V., 90.

Wangerin, A., 332; Watanabe, M., 282; Watson, G. N., 283; Webster, A. G., 235; Wedderburn, J. H. M., 91; Weld, L. G., 237, 261; Wells, M. E., 141; Wellstein, J., 188; Wester, C. W., 467; White, H. S., 425; Whittemore, J. K., 283; Whyte, W. A., 333; Wieleitner, H., 332; Wilczynski, E. J., 234; Williams, K. P., 283; Williams, W. L. G., 333, 379; Wilson, N. R., 283; Wilson, W. H., 284; Wolfe, C. L. E., 137; Wolfe, H. E., 137, 235; Wolff, H. C., 91; Wood, F., 187; Woods, B. M., 186; Woodward, R. S., 467.

Yeaton, C. H., 141; Young, J. W., 234; Young, W. H., 90.

Zeldin, S. D., 93; Zeuthen, H. G., 284.

Prizes:

Astronomical Society of the Pacific, 283; Barnard Medal, 468; Belgian Academy of Sciences, 138; Belgian Government, 138; Göttingen Society of Sciences, 331; Nobel, 466; Paris Academy of Sciences, 138, 282; Peter Wilhelm Müller Foundation, 332; Prince Jablonowski Society, 332; Querini-Stampalia, 91; Royal Society of London, 185; Smith, 379.

Universities and Technical Schools.

Bologna, 87.
California, 140, 428.
Catania, 88.
Chicago, 331, 380.
Columbia, 378, 426.
Cornell, 140, 427.
Genoa, 88.
Harvard, 184, 427.
Massachusetts Institute of
Technology, 428.
Messina, 88.
Naples, 88.

Padua, 88.
Palermo, 89.
Pavia, 89.
Pisa, 89.
Princeton, 465.
Rome, 89.
Stanford, 465.
Strasbourg, 137, 429.
Turin, 90.
Wisconsin, 428.
Yale, 379.

Whole No. 290

CONTENTS

	PAGE
The April Meeting of the American Mathematical Society in New York. By Professor F. N. COLE - - -	433
Stieltjes Derivatives. By Professor P. J. DANIELL - -	444
Sheffer's Set of Five Postulates for Boolean Algebras in Terms of the Operation "Rejection" Made Completely Independent. By Dr. J. S. TAYLOR - - -	449
Rotation Surfaces of Constant Curvature in Space of Four Dimensions. By Professor C. L. E. MOORE - -	454
Shorter Notices - - - - -	460
Notes - - - - -	464
New Publications - - - - -	469
Twenty-Ninth Annual List of Published Papers - -	472
Index of Volume Twenty-Six - - - - -	480

Changes of address of members, exchanges, and subscribers should be communicated at once to the Secretary of the American Mathematical Society, 501 West 116th Street, New York.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to The American Mathematical Society, 41 North Queen St., Lancaster, Pa., or 501 West 116th Street, New York.

The initiation fees and annual dues of members of the American Mathematical Society are payable to the Treasurer of the American Mathematical Society, Professor J. H. TANNER, Cornell University, Ithaca, N. Y.

Articles for insertion in the BULLETIN should be addressed to the
BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.

The following dates have been fixed for the meetings of the Society :

The Twenty-Seventh Summer Meeting and Ninth Colloquium of the Society will be held at the University of Chicago, extending through the week September 7-11, 1920.

